

DIAGONALIZATION OF AN INTEGRABLE DISCRETIZATION OF THE REPULSIVE DELTA BOSE GAS ON THE CIRCLE

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ABSTRACT. We introduce an integrable lattice discretization of the quantum system of n bosonic particles on a ring interacting pairwise via repulsive delta potentials. The corresponding (finite-dimensional) spectral problem of the integrable lattice model is solved by means of the Bethe Ansatz method. The resulting eigenfunctions turn out to be given by specializations of the Hall-Littlewood polynomials. In the continuum limit the solution of the repulsive delta Bose gas due to Lieb and Liniger is recovered, including the orthogonality of the Bethe wave functions first proved by Dorlas (extending previous work of C.N. Yang and C.P. Yang).

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1. INTRODUCTION

The non-ideal Bose gas with delta-potential interactions is a system of n one-dimensional bosonic particles characterized by a Hamiltonian given by the formal Schrödinger operator

$$H = -\Delta + g \sum_{1 \leq j \neq k \leq n} \delta(x_j - x_k). \quad (1.1)$$

Here x_1, \dots, x_n represent the position variables, $\Delta := \partial_{x_1}^2 + \dots + \partial_{x_n}^2$, δ refers to the delta distribution, and g denotes a coupling parameter determining the strength of the interaction. For $g > 0$ the interaction between the particles is repulsive and for $g < 0$ it is attractive, whereas for $g = 0$ the model degenerates to an ideal one-dimensional boson gas without interaction between the particles.

The eigenvalue problem for the above Schrödinger operator with periodic boundary conditions—i.e. for particles moving along a circle—was solved by Lieb and Liniger by means of the Bethe Ansatz method [LL]. The corresponding spectral problem for particles moving along the whole real line was considered subsequently by McGuire [Mc]. Since the appearance of these two pioneering papers, the exactly solvable quantum models under consideration have been the subject of numerous studies; for an overview of the vast literature and an extensive bibliography we refer the reader to Refs. [Ma, G4, KBI, AK, S]. Further generalizations of the one-dimensional quantum n -particle system with delta-potential interactions can be found in Refs. [G3, Gu, HO, Di], where analogous quantum eigenvalue problems are studied in which the permutation-symmetry is traded for an invariance with respect to the action of more general reflection groups [B, Hu], and also in Refs. [AFK, AK, CC, HLP], where the delta-potential interaction between the particles is replaced by more general zero-range point-like interactions (involving combinations of δ and δ' type potentials) [A-H].

An important (and notoriously hard) problem connected with the Bethe Ansatz method is the question of demonstrating the completeness of the Bethe wave functions in a Hilbert space context. For the non-ideal Bose gas on the line with a repulsive delta-potential interaction ($g > 0$), the spectrum of the Schrödinger operator is purely continuous. The completeness of the Bethe wave functions was proved for this case by Gaudin [G1, G2]. For the corresponding system in the attractive regime ($g < 0$), the completeness problem is much harder as multi-particle binding may occur thus giving rise to mixed continuous-discrete spectrum. In this more complex situation the completeness of the Bethe wave functions was shown by Oxford [O], with the aid of techniques developed by Babbitt and Thomas in their treatment of an analogous spectral problem for the one-dimensional infinite isotropic Heisenberg spin chain [T, BT]. When passing from particles on the line to particles on the circle the nature of the system changes drastically, as the confinement to a compact region forces the spectrum of the Schrödinger operator to become purely discrete. In this situation the completeness of the Bethe wave functions was proved for the repulsive regime by Dorlas [Do]. In Dorlas' approach the question of the completeness is first reduced to that of the orthogonality of the Bethe wave functions. This orthogonality is then shown to hold with the aid of quantum inverse scattering theory [KBI], combined with previous results of C.N. Yang and C.P. Yang pertaining to the solution of the associated algebraic system of Bethe equations (determining the spectrum of the Schrödinger operator under

consideration) [YY]. For the attractive regime such progress has yet to be made: the question of the construction of a complete eigenbasis for the non-ideal Bose gas on the circle with delta-potential interaction remains (to date) open.

By exploiting the translational invariance and the permutation symmetry the eigenvalue problem characterized by the Hamiltonian H (1.1) reduces—in the case of bosonic particles moving along a circle of unit circumference—to that of the free Laplacian

$$-\Delta\psi = E\psi, \quad (1.2)$$

acting on a domain of wave functions $\psi(x_1, \dots, x_n)$ with support inside the alcove

$$\mathbf{A} = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0, x_1 \geq x_2 \geq \dots \geq x_n, x_1 - x_n \leq 1\}, \quad (1.3)$$

and subject to normal linear homogeneous boundary conditions at the walls of the alcove of the form

$$(\partial_{x_j} - \partial_{x_{j+1}} - g)\psi|_{x_j - x_{j+1} = 0} = 0, \quad j = 1, \dots, n-1, \quad (1.4a)$$

$$(\partial_{x_n} - \partial_{x_1} - g)\psi|_{x_1 - x_n = 1} = 0. \quad (1.4b)$$

(The parameter E represents the energy eigenvalue.) The purpose of the present paper is to study a discretization of the eigenvalue problem in Eqs. (1.2)–(1.4b). Throughout we will restrict attention to the repulsive parameter regime $g > 0$.

More specifically, we study the eigenvalue problem for an integrable system of discrete Laplacians acting on functions supported on a regular lattice over the alcove \mathbf{A} (1.3), and subject to repulsive reflection relations at the boundary of the lattice. Since the alcove is compact, the lattice in question is finite; hence, our discrete eigenvalue problem is finite-dimensional. We solve the eigenvalue problem at issue by means of the Bethe Ansatz method. The resulting Bethe eigenfunctions turn out to be given by specializations of the Hall-Littlewood polynomials [M2, M3]. The orthogonality and completeness of these Bethe eigenfunctions arises as an immediate consequence of the integrability (which permits removing possible degeneracies in the spectrum of the Laplacians). As a byproduct, the Lieb-Liniger type Bethe eigenfunctions for the eigenvalue problem in Eqs. (1.2)–(1.4b) are recovered via a continuum limit. The orthogonality of the latter eigenfunctions (and thus eventually—because of Dorlas’ results [Do]—also the completeness) are in our approach immediately inherited from the corresponding orthogonality results for our discretized lattice model. In this connection it is probably helpful to recall that the original proof of the orthogonality due to Dorlas [Do] also involves a discretization, which arises however in a fundamentally different way from the one employed here. In a nutshell: Dorlas arrives at the orthogonality through a continuum limit of the (second) quantization of the Lattice Nonlinear Schrödinger Equation introduced by Izergin and Korepin [KBI], whereas here—in contrast—we study a rather more elementary quantum lattice model characterized by a direct discretization of the Schrödinger operator in Eqs. (1.2)–(1.4b) itself.

The paper is structured as follows. In Section 2 the discretization of the eigenvalue problem in Eqs. (1.2)–(1.4b) is formulated. In Section 3 the eigenfunctions are constructed by means of the Bethe Ansatz method. The associated Bethe equations are solved in Section 4 and the orthogonality and completeness of the corresponding Bethe wave functions is demonstrated in Section 5. Finally, the continuum limit is analyzed in Section 6.

2. DISCRETE LAPLACIANS ON THE ALCOVE

In this section we introduce a system of discrete Laplacians on a finite lattice over (a dilated version of) the alcove \mathbf{A} (1.3). For this purpose it will be convenient to borrow concepts and notation from the theory of root systems. Here we will only need to deal with the simplest type of root systems: those of type A . For further background material concerning root systems the reader is referred to Refs. [B, Hu].

2.1. Preliminaries. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the standard basis of unit vectors in \mathbb{R}^n and let $\langle \cdot, \cdot \rangle$ be the (usual) inner product with respect to which the standard basis is orthonormal. The alcove \mathbf{A} (1.3) constitutes a convex polyhedron in the center-of-mass plane

$$\mathbf{E} = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{e} \rangle = 0\}, \quad \mathbf{e} = \mathbf{e}_1 + \dots + \mathbf{e}_n, \quad (2.1)$$

which is bounded by the n hyperplanes

$$\mathbf{E}_0 = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \boldsymbol{\alpha}_0 \rangle = 1\}, \quad (2.2a)$$

$$\mathbf{E}_j = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \boldsymbol{\alpha}_j \rangle = 0\}, \quad j = 1, \dots, n-1, \quad (2.2b)$$

where

$$\boldsymbol{\alpha}_0 := \mathbf{e}_1 - \mathbf{e}_n \quad \text{and} \quad \boldsymbol{\alpha}_j := \mathbf{e}_j - \mathbf{e}_{j+1}, \quad j = 1, \dots, n-1. \quad (2.3)$$

Specifically, we have that

$$\mathbf{A} = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \boldsymbol{\alpha}_0 \rangle \leq 1, \langle \mathbf{x}, \boldsymbol{\alpha}_j \rangle \geq 0, j = 1, \dots, n-1\}. \quad (2.4)$$

The vertices (i.e. corners) of the polyhedron \mathbf{A} are determined by the intersections of all choices of $n-1 (= \dim(\mathbf{E}))$ out of the n hyperplanes $\mathbf{E}_0, \dots, \mathbf{E}_{n-1}$. These vertices are given explicitly by the origin $\mathbf{0}$ and the vectors

$$\boldsymbol{\omega}_j := \mathbf{e}_1 + \dots + \mathbf{e}_j - \frac{j}{n}(\mathbf{e}_1 + \dots + \mathbf{e}_n), \quad j = 1, \dots, n-1. \quad (2.5)$$

Indeed, the vectors $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{n-1}$ all lie on the hyperplane \mathbf{E}_0 (2.2a) and constitute a basis of \mathbf{E} that is dual to the basis $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1}$ (in the sense that $\langle \boldsymbol{\omega}_j, \boldsymbol{\alpha}_k \rangle = \delta_{j,k}$, where $\delta_{j,k}$ denotes the Kronecker delta symbol).

Let $r_0 : \mathbf{E} \rightarrow \mathbf{E}$ be the orthogonal reflection in the hyperplane \mathbf{E}_0 (2.2a) and let $r_j : \mathbf{E} \rightarrow \mathbf{E}$, $j = 1, \dots, n-1$ be the orthogonal reflections in the hyperplanes \mathbf{E}_j (2.2b). The action of these reflections on an arbitrary vector $\mathbf{x} \in \mathbf{E}$ is of the form

$$r_0(\mathbf{x}) = \mathbf{x} + (1 - \langle \mathbf{x}, \boldsymbol{\alpha}_0 \rangle) \boldsymbol{\alpha}_0, \quad (2.6a)$$

$$r_j(\mathbf{x}) = \mathbf{x} - \langle \mathbf{x}, \boldsymbol{\alpha}_j \rangle \boldsymbol{\alpha}_j, \quad j = 1, \dots, n-1. \quad (2.6b)$$

From these two formulas it is readily inferred that for $j \in \{1, \dots, n-1\}$ the reflection r_j swaps the j^{th} and $(j+1)^{\text{th}}$ coordinates of \mathbf{x} , and that r_0 swaps the first and the last coordinates followed by a translation over the vector $\boldsymbol{\alpha}_0$. Hence, the reflections r_1, \dots, r_{n-1} generate an action of the permutation group \mathcal{S}_n on \mathbf{E} , and the reflections r_0, \dots, r_{n-1} generate an action of the affine permutation group $\hat{\mathcal{S}}_n = \mathcal{S}_n \ltimes \mathcal{Q}$, which is the semidirect product of the permutation group \mathcal{S}_n and the lattice of translations

$$\mathcal{Q} := \text{Span}_{\mathbb{Z}}(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-1}). \quad (2.7)$$

For any (affine) permutation $\sigma \in \hat{\mathcal{S}}_n$, one defines its *length* $\ell(\sigma)$ as the minimal number of reflections needed for decomposing σ (non-uniquely) in terms of the generators:

$$\sigma = r_{j_1} r_{j_2} \cdots r_{j_\ell} \quad (2.8)$$

(where $j_1, \dots, j_\ell \in \{0, 1, \dots, n-1\}$ and with the convention that the length of the identity element is equal to zero).

The polyhedron \mathbf{A} (2.4) constitutes a fundamental domain for the action of $\hat{\mathcal{S}}_n$ on \mathbf{E} . More specifically, for each $\mathbf{x} \in \mathbf{E}$ the orbit $\hat{\mathcal{S}}_n(\mathbf{x})$ intersects \mathbf{A} precisely once. Let us denote by $\sigma_{\mathbf{x}} \in \hat{\mathcal{S}}_n$ the unique shortest affine permutation such that

$$\sigma_{\mathbf{x}}(\mathbf{x}) \in \mathbf{A}. \quad (2.9)$$

Let us fix a positive integer m . Below it will often be convenient to employ a dilated version of the polyhedron $\mathbf{A}^{(m)} := m\mathbf{A}$ rather than \mathbf{A} itself. Throughout we shall distinguish by means of a superscript (m) the corresponding boundary planes, boundary reflections, and the elements of the affine permutation group $\hat{\mathcal{S}}_n^{(m)} := \mathcal{S}_n \ltimes (m\mathcal{Q}) \subset \hat{\mathcal{S}}_n$ generated by these reflections. That is to say, the dilated alcove $\mathbf{A}^{(m)}$ is bounded by the hyperplanes $\mathbf{E}_j^{(m)} := m\mathbf{E}_j$, $j = 0, \dots, n-1$; the orthogonal reflections in these hyperplanes act as $r_j^{(m)}(\mathbf{x}) := mr_j(\mathbf{x}/m)$, $j = 0, \dots, n-1$. (So $\mathbf{E}_j^{(m)} = \mathbf{E}_j$ and $r_j^{(m)} = r_j$ if $j > 0$.) The affine permutation $\sigma_{\mathbf{x}}^{(m)} \in \hat{\mathcal{S}}_n^{(m)}$ mapping a vector $\mathbf{x} \in \mathbf{E}$ into the fundamental domain $\mathbf{A}^{(m)}$ is given by the action $\sigma_{\mathbf{x}}^{(m)}(\mathbf{x}) := m\sigma_{\mathbf{x}/m}(\mathbf{x}/m)$.

2.2. Laplacians. The orthogonal projection of $\mathbb{Z}^n \subset \mathbb{R}^n$ onto the center-of-mass plane \mathbf{E} (2.1) is given by the lattice dual to \mathcal{Q} (2.7):

$$\mathcal{P} := \{\boldsymbol{\lambda} \in \mathbf{E} \mid \forall \boldsymbol{\alpha} \in \mathcal{Q}: \langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle \in \mathbb{Z}\} \quad (2.10a)$$

$$= \text{Span}_{\mathbb{Z}}(\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{n-1}). \quad (2.10b)$$

It is clear from Eq. (2.10a) that \mathcal{Q} is contained as a sublattice in \mathcal{P} . Hence, the action of the affine permutation group $\hat{\mathcal{S}}_n$ in \mathbf{E} maps the lattice \mathcal{P} into itself (cf. Eqs. (2.6a), (2.6b)). The intersection of the lattice \mathcal{P} with the dilated polyhedron $\mathbf{A}^{(m)}$ provides a finite grid $\mathcal{P}^{(m)}$ over $\mathbf{A}^{(m)}$ containing its vertices $\mathbf{0}$ and $m\boldsymbol{\omega}_1, \dots, m\boldsymbol{\omega}_{n-1}$:

$$\begin{aligned} \mathcal{P}^{(m)} &:= \{\boldsymbol{\lambda} \in \mathcal{P} \mid \boldsymbol{\lambda} \in \mathbf{A}^{(m)}\} \\ &= \{k_1\boldsymbol{\omega}_1 + \dots + k_{n-1}\boldsymbol{\omega}_{n-1} \mid k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 0}, k_1 + \dots + k_{n-1} \leq m\}. \end{aligned} \quad (2.11)$$

We are now in the position to define a system of $n-1$ Laplace operators acting in the space $\mathcal{C}(\mathcal{P}^{(m)})$ of complex functions $\psi : \mathcal{P}^{(m)} \rightarrow \mathbb{C}$.

Definition (Laplace Operator). To each basis vector $\boldsymbol{\omega}_k$ from Eq. (2.5) we associate a corresponding Laplace operator $L_k^{(m)} : \mathcal{C}(\mathcal{P}^{(m)}) \rightarrow \mathcal{C}(\mathcal{P}^{(m)})$ defined by its action on an arbitrary function $\psi : \mathcal{P}^{(m)} \rightarrow \mathbb{C}$ of the form

$$(L_k^{(m)}\psi)_{\boldsymbol{\lambda}} := \sum_{\boldsymbol{\nu} \in \mathcal{S}_n(\boldsymbol{\omega}_k)} \psi_{\boldsymbol{\lambda}+\boldsymbol{\nu}}, \quad (2.12a)$$

with the boundary convention that for $\boldsymbol{\mu} \in \mathcal{P} \setminus \mathcal{P}^{(m)}$

$$\psi_{\boldsymbol{\mu}} := t^{\ell^{(m)}(\sigma_{\boldsymbol{\mu}}^{(m)})} \psi_{\sigma_{\boldsymbol{\mu}}^{(m)}(\boldsymbol{\mu})}, \quad (2.12b)$$

where t denotes a real (coupling) parameter (and the length function $\ell^{(m)}(\cdot)$ refers to the minimal number of reflections in the decomposition of an affine permutation in $\hat{\mathcal{S}}_n^{(m)}$ in terms of $r_0^{(m)}, \dots, r_{n-1}^{(m)}$).

Roughly speaking, the value of $L_k^{(m)}\psi$ in a point $\lambda \in \mathcal{P}^{(m)}$ is equal to the sum of the values of ψ in all neighboring points of the form $\lambda + \nu$, where ν runs through the orbit of ω_k with respect to the action of the permutation group \mathcal{S}_n . When $\lambda + \nu$ lies outside $\mathcal{P}^{(m)}$ the value of $\psi_{\lambda+\nu}$ is governed by the boundary convention in Eq. (2.12b). It is instructive to clarify the nature of this boundary convention somewhat more in detail by decomposing $\sigma_{\lambda+\nu}^{(m)}$ in terms of the elementary reflections in the hyperplanes bounding $\mathbf{A}^{(m)}$.

Proposition 2.1 (Boundary Reflection Relations). *Let $\lambda \in \mathcal{P}^{(m)}$. The boundary convention in Eq. (2.12b) amounts to the requirement that $\forall \nu \in \mathcal{S}_n(\omega_k)$ for which $\lambda + \nu \in \mathcal{P} \setminus \mathcal{P}^{(m)}$*

$$\psi_{\lambda+\nu} = \begin{cases} t\psi_{r_0^{(m)}(\lambda+\nu)} & \text{if } \langle \lambda + \nu, \alpha_0 \rangle > m \\ t\psi_{r_j^{(m)}(\lambda+\nu)} & \text{if } \langle \lambda + \nu, \alpha_j \rangle < 0 \quad (j > 0), \end{cases} \quad (2.13a)$$

or equivalently

$$\psi_{\lambda+\nu} = \begin{cases} t\psi_{\lambda+\nu-\alpha_0} & \text{if } \langle \lambda, \alpha_0 \rangle = m \text{ and } \langle \nu, \alpha_0 \rangle = 1 \\ t\psi_{\lambda+\nu+\alpha_j} & \text{if } \langle \lambda, \alpha_j \rangle = 0 \text{ and } \langle \nu, \alpha_j \rangle = -1 \quad (j > 0). \end{cases} \quad (2.13b)$$

Proof. Let $\lambda \in \mathcal{P}^{(m)}$ and $\nu \in \mathcal{S}_n(\omega_k)$ such that $\lambda + \nu \in \mathcal{P} \setminus \mathcal{P}^{(m)}$. Then there exist $j \in \{0, \dots, n-1\}$ such that $\langle \lambda + \nu, \alpha_j \rangle$ is either $> m$ if $j = 0$ or < 0 if $j > 0$. Geometrically, this means that the hyperplane $\mathbf{E}_j^{(m)}$ separates $\lambda + \nu$ from $\sigma_{\lambda+\nu}^{(m)}(\lambda + \nu) \in \mathcal{P}^{(m)}$. Let us write $\mu := r_j^{(m)}(\lambda + \nu)$. Then for any such j we have that

$$\sigma_{\lambda+\nu}^{(m)} = \sigma_\mu^{(m)} r_j^{(m)} \quad \text{with} \quad \ell^{(m)}(\sigma_\mu^{(m)}) = \ell^{(m)}(\sigma_{\lambda+\nu}^{(m)}) - 1.$$

We will now use induction on the length of $\sigma_{\lambda+\nu}^{(m)}$ to prove that the boundary convention in Eq. (2.12b) and the boundary reflection relation in Eq. (2.13a) are equivalent. Indeed, upon assuming (2.12b) it is clear that

$$\psi_{\lambda+\nu} = t^{\ell^{(m)}(\sigma_{\lambda+\nu}^{(m)})} \psi_{\sigma_{\lambda+\nu}^{(m)}(\lambda+\nu)} = t^{\ell^{(m)}(\sigma_\mu^{(m)})+1} \psi_{\sigma_\mu^{(m)}(\mu)} = t \psi_\mu,$$

which amounts to (2.13a). Reversely, upon assuming (2.13a) and invoking the induction hypothesis we see that

$$\psi_{\lambda+\nu} = t \psi_\mu = t^{\ell^{(m)}(\sigma_\mu^{(m)})+1} \psi_{\sigma_\mu^{(m)}(\mu)} = t^{\ell^{(m)}(\sigma_{\lambda+\nu}^{(m)})} \psi_{\sigma_{\lambda+\nu}^{(m)}(\lambda+\nu)},$$

which amounts to (2.12b). To finish the proof of the proposition it remains to check that the boundary reflection relations in Eqs. (2.13a) and (2.13b) are equivalent. For this purpose it suffices to notice that the requirements that $\lambda \in \mathcal{P}^{(m)}$ and $\nu \in \mathcal{S}_n(\omega_k)$ imply that $0 \leq \langle \lambda, \alpha_j \rangle \leq m$ and that $-1 \leq \langle \nu, \alpha_j \rangle \leq 1$ for $j = 0, \dots, n-1$. Hence $\langle \lambda + \nu, \alpha_0 \rangle > m$ iff $\langle \lambda, \alpha_0 \rangle = m$ and $\langle \nu, \alpha_0 \rangle = 1$, in which case $r_0^{(m)}(\lambda + \nu) = \lambda + \nu - \alpha_0$, and furthermore for $j > 0$ one has that $\langle \lambda + \nu, \alpha_j \rangle < 0$ iff $\langle \lambda, \alpha_j \rangle = 0$ and $\langle \nu, \alpha_j \rangle = -1$, in which case $r_j^{(m)}(\lambda + \nu) = \lambda + \nu + \alpha_j$. \square

It is clear from the proposition that the boundary convention in Eq. (2.12b) amounts to a normal linear boundary condition at the hyperplanes bounding $\mathbf{A}^{(m)}$. The coupling parameter t determines the nature of this boundary condition; for $|t| > 1$ the boundary term (i.e. the interaction between the particles) is attractive

whereas for $|t| < 1$ it is repulsive. For $t = 0$ and $t = 1$ we are dealing with Dirichlet type and Neumann type boundary conditions, respectively.

Applying the boundary convention to those contributions in the sum over the translated orbit $\lambda + \mathcal{S}_n(\omega_k)$ corresponding to lattice points outside the grid $\mathcal{P}^{(m)}$ gives rise to a closed formula for the action of the Laplace operator in which the value of $L_k^{(m)}\psi$ at the point λ is expressed completely in terms of the values of ψ at the neighboring points of the form $\lambda + \nu \in \mathcal{P}^{(m)}$. To make this procedure completely explicit we shall need some further notation. Let \mathbf{R} be the orbit of the basis $\alpha_1, \dots, \alpha_{n-1}$ with respect to the action of the permutation group \mathcal{S}_n and let \mathbf{R}^+ be the part of the orbit that expands nonnegatively with respect to this basis:

$$\mathbf{R} = \{\mathbf{e}_j - \mathbf{e}_k \mid 1 \leq j \neq k \leq n\}, \quad \mathbf{R}^+ = \{\mathbf{e}_j - \mathbf{e}_k \mid 1 \leq j < k \leq n\}. \quad (2.14)$$

Proposition 2.2 (Explicit Action of the Laplace Operator). *The action of the Laplace operator $L_k^{(m)}$ ($k \in \{1, \dots, n-1\}$) on an arbitrary grid function $\psi : \mathcal{P}^{(m)} \rightarrow \mathbb{C}$ is of the form*

$$(L_k^{(m)}\psi)_\lambda = \sum_{\substack{\nu \in \mathcal{S}_n(\omega_k) \\ \lambda + \nu \in \mathcal{P}^{(m)}}} V_{\lambda, \nu}^{(m)} \psi_{\lambda + \nu}, \quad (2.15a)$$

where

$$V_{\lambda, \nu}^{(m)} := \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = 0 \\ \langle \nu, \alpha \rangle = 1}} \frac{1 - t^{1 + \langle \rho, \alpha \rangle}}{1 - t^{\langle \rho, \alpha \rangle}} \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = m \\ \langle \nu, \alpha \rangle = -1}} \frac{1 - t^{1 + n - \langle \rho, \alpha \rangle}}{1 - t^{n - \langle \rho, \alpha \rangle}}, \quad (2.15b)$$

with $\rho := \sum_{\alpha \in \mathbf{R}^+} \alpha / 2 = \omega_1 + \dots + \omega_{n-1}$.

Proof. It follows from (the proof of) Proposition 2.1 that for $\lambda \in \mathcal{P}^{(m)}$ and $\nu \in \mathcal{S}_n(\omega_k)$ such that $\lambda + \nu \in \mathcal{P}^{(m)}$ the coefficient $V_{\lambda, \nu}^{(m)}$ of $\psi_{\lambda + \nu}$ in $(L_k^{(m)}\psi)_\lambda$ is of the form $\sum_{\sigma} t^{\ell^{(m)}(\sigma)}$, where the sum is over those affine permutations σ in $\hat{\mathcal{S}}_n^{(m)}$ that are generated by iterated application of reflections of the type figuring in Eqs. (2.13a), (2.13b). These affine permutations are precisely the $\sigma \in \hat{\mathcal{S}}_n^{(m)}$ for which $\sigma(\lambda) = \lambda$ and $\sigma(\lambda + \nu) \notin \mathcal{P}^{(m)}$, or equivalently, $\sigma(\lambda + \nu) \neq \lambda + \nu$. In other words, the coefficient is equal to the Poincaré series of the quotient of the stabilizer subgroup $\hat{\mathcal{S}}_{n, \lambda}^{(m)} := \{\sigma \in \hat{\mathcal{S}}_n^{(m)} \mid \sigma(\lambda) = \lambda\}$ and the stabilizer subgroup $\hat{\mathcal{S}}_{n, \lambda}^{(m)} \cap \hat{\mathcal{S}}_{n, \lambda + \nu}^{(m)}$, i.e.

$$V_{\lambda, \nu}^{(m)} = \sum_{\sigma \in \hat{\mathcal{S}}_{n, \lambda}^{(m)} / (\hat{\mathcal{S}}_{n, \lambda}^{(m)} \cap \hat{\mathcal{S}}_{n, \lambda + \nu}^{(m)})} t^{\ell^{(m)}(\sigma)} = \frac{\sum_{\sigma \in \hat{\mathcal{S}}_{n, \lambda}^{(m)}} t^{\ell^{(m)}(\sigma)}}{\sum_{\sigma \in \hat{\mathcal{S}}_{n, \lambda}^{(m)} \cap \hat{\mathcal{S}}_{n, \lambda + \nu}^{(m)}} t^{\ell^{(m)}(\sigma)}}. \quad (2.16)$$

It follows from a general formula for the Poincaré series of (affine) Weyl groups due to Macdonald [M1] (cf. Corollaries (2.5) and (3.4)) that the Poincaré series in the numerator and denominator of Eq. (2.16) admit product representations given by

$$\sum_{\sigma \in \hat{\mathcal{S}}_{n, \lambda}^{(m)}} t^{\ell^{(m)}(\sigma)} = \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = 0}} \frac{1 - t^{1 + \langle \rho, \alpha \rangle}}{1 - t^{\langle \rho, \alpha \rangle}} \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = m}} \frac{1 - t^{1 + n - \langle \rho, \alpha \rangle}}{1 - t^{n - \langle \rho, \alpha \rangle}} \quad (2.17)$$

and

$$\sum_{\sigma \in \hat{\mathcal{S}}_{n,\lambda}^{(m)} \cap \hat{\mathcal{S}}_{n,\lambda+\nu}^{(m)}} t^{\ell^{(m)}(\sigma)} = \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = 0 \\ \langle \lambda + \nu, \alpha \rangle = 0}} \frac{1 - t^{1+\langle \rho, \alpha \rangle}}{1 - t^{\langle \rho, \alpha \rangle}} \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = m \\ \langle \lambda + \nu, \alpha \rangle = m}} \frac{1 - t^{1+n-\langle \rho, \alpha \rangle}}{1 - t^{n-\langle \rho, \alpha \rangle}}, \quad (2.18)$$

respectively, which—upon inserting in Eq. (2.16)—gives rise to Eq. (2.15b). \square

Remark. The stabilizer subgroups $\hat{\mathcal{S}}_{n,\lambda}^{(m)}$ and $\hat{\mathcal{S}}_{n,\lambda}^{(m)} \cap \hat{\mathcal{S}}_{n,\lambda+\nu}^{(m)}$ in the proof of Proposition 2.2 consist of (direct products of) permutation groups. It is well-known (and readily seen by induction) that the Poincaré series of the permutation group \mathcal{S}_ℓ admits the product representation $\prod_{j=1}^\ell (1 - t^j)/(1 - t)$ (cf. e.g. Ref. [M2], Chapter III, §1). With the aid of this latter product formula it is not so difficult to verify Eqs. (2.17), (2.18) (and thus Eq. (2.15b)) directly (i.e. without invoking the much more general results of Macdonald [M1] cited in the proof). (We thank the referee for making this point.)

2.3. Hilbert Space Structure. We shall now endow the function space $\mathcal{C}(\mathcal{P}^{(m)})$ with a inner product, turning it into a (finite-dimensional) Hilbert space $\mathcal{H}^{(m)} := \ell^2(\mathcal{P}^{(m)}, \Delta^{(m)})$ characterized by a positive weight function $\Delta^{(m)} : \mathcal{P}^{(m)} \rightarrow (0, \infty)$. To this end it will always be assumed from here onwards that the coupling parameter t lies in the repulsive regime

$$-1 < t < 1 \quad (2.19)$$

(unless explicitly stated otherwise). For two arbitrary functions $\psi, \phi \in \mathcal{C}(\mathcal{P}^{(m)})$ the inner product in question is then defined as

$$\langle \psi, \phi \rangle^{(m)} := \sum_{\lambda \in \mathcal{P}^{(m)}} \psi_\lambda \overline{\phi_\lambda} \Delta_\lambda^{(m)}, \quad (2.20a)$$

where the weight function is given by

$$\Delta_\lambda^{(m)} := \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = 0}} \frac{1 - t^{\langle \rho, \alpha \rangle}}{1 - t^{1+\langle \rho, \alpha \rangle}} \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \lambda, \alpha \rangle = m}} \frac{1 - t^{n-\langle \rho, \alpha \rangle}}{1 - t^{1+n-\langle \rho, \alpha \rangle}}. \quad (2.20b)$$

(Notice in this connection that the restriction of the coupling parameter to the repulsive regime (2.19) ensures that the values of the weight function $\Delta_\lambda^{(m)}$ are indeed positive for all $\lambda \in \mathcal{P}^{(m)}$.)

Proposition 2.3 (Adjoint). *The Laplace operators $L_k^{(m)}$ and $L_{n-k}^{(m)}$ ($k \in \{1, \dots, n-1\}$) are each others adjoints in $\mathcal{H}^{(m)}$*

$$\forall \psi, \phi \in \mathcal{C}(\mathcal{P}^{(m)}) : \quad \langle L_k^{(m)} \psi, \phi \rangle^{(m)} = \langle \psi, L_{n-k}^{(m)} \phi \rangle^{(m)}. \quad (2.21)$$

Proof. The proof hinges on the explicit formula for the action of $L_k^{(m)}$ in Proposition 2.2. Elementary manipulations reveal that

$$\begin{aligned}
\langle L_k^{(m)} \psi, \phi \rangle^{(m)} &= \sum_{\lambda \in \mathcal{P}^{(m)}} (L_k^{(m)} \psi)_\lambda \overline{\phi_\lambda} \Delta_\lambda^{(m)} \\
&= \sum_{\nu \in \mathcal{S}_n(\omega_k)} \sum_{\substack{\lambda \in \mathcal{P}^{(m)} \\ \lambda + \nu \in \mathcal{P}^{(m)}}} V_{\lambda, \nu}^{(m)} \psi_{\lambda + \nu} \overline{\phi_\lambda} \Delta_\lambda^{(m)} \\
&\stackrel{(i)}{=} \sum_{\nu \in \mathcal{S}_n(\omega_k)} \sum_{\substack{\mu \in \mathcal{P}^{(m)} \\ \mu - \nu \in \mathcal{P}^{(m)}}} \psi_\mu V_{\mu - \nu, \nu}^{(m)} \overline{\phi_{\mu - \nu}} \Delta_{\mu - \nu}^{(m)} \\
&\stackrel{(ii)}{=} \sum_{\nu \in \mathcal{S}_n(\omega_k)} \sum_{\substack{\mu \in \mathcal{P}^{(m)} \\ \mu - \nu \in \mathcal{P}^{(m)}}} \psi_\mu V_{\mu, -\nu}^{(m)} \overline{\phi_{\mu - \nu}} \Delta_\mu^{(m)} \\
&\stackrel{(iii)}{=} \sum_{\mu \in \mathcal{P}^{(m)}} \psi_\mu \overline{(L_{n-k}^{(m)} \phi)_\mu} \Delta_\mu^{(m)} = \langle \psi, L_{n-k}^{(m)} \phi \rangle^{(m)},
\end{aligned}$$

where we have used: (i) the substitution $\lambda = \mu - \nu$, (ii) the identity

$$V_{\mu - \nu, \nu}^{(m)} \Delta_{\mu - \nu}^{(m)} = V_{\mu, -\nu}^{(m)} \Delta_\mu^{(m)}, \quad (2.22)$$

and (iii) the facts that the coefficient $V_{\mu, -\nu}^{(m)}$ is real and $-\omega_k \in \mathcal{S}_n(\omega_{n-k})$. To infer the identity in Eq. (2.22) one observes that for $\mu \in \mathcal{P}^{(m)}$ and $\nu \in \mathcal{S}_n(\omega_k)$ such that $\mu - \nu \in \mathcal{P}^{(m)}$ both sides $V_{\mu - \nu, \nu}^{(m)} \Delta_{\mu - \nu}^{(m)}$ and $V_{\mu, -\nu}^{(m)} \Delta_\mu^{(m)}$ reduce—upon canceling common terms from the numerator and denominator—to

$$\prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \mu, \alpha \rangle = 0 \\ \langle \nu, \alpha \rangle = 0}} \frac{1 - t^{\langle \rho, \alpha \rangle}}{1 - t^{1 + \langle \rho, \alpha \rangle}} \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \mu, \alpha \rangle = m \\ \langle \nu, \alpha \rangle = 0}} \frac{1 - t^{n - \langle \rho, \alpha \rangle}}{1 - t^{1 + n - \langle \rho, \alpha \rangle}}.$$

□

3. BETHE ANSATZ EIGENFUNCTIONS

In this section the eigenfunctions of the Laplacian $L_k^{(m)}$ (2.12a), (2.12b) are constructed by means of the Bethe Ansatz method of Lieb and Liniger [LL, Ma, G4, KBI].

3.1. Bethe Ansatz. If we ignore boundary effects for a moment and interpret $L_k^{(m)}$ (2.12a) (*without* the boundary convention (2.12b)) as a Laplacian acting on functions $\psi : \mathcal{P} \rightarrow \mathbb{C}$, then clearly the plane wave $\psi_\lambda(\xi) = \exp(i\langle \lambda, \xi \rangle)$ with wave number $\xi \in \mathbf{E}/(2\pi\mathcal{Q})$ constitutes an eigenfunction corresponding to the eigenvalue $E_k(\xi) = \sum_{\nu \in \mathcal{S}_n(\omega_k)} \exp(i\langle \nu, \xi \rangle)$. The Bethe Ansatz method aims to construct the eigenfunctions $\psi : \mathcal{P}^{(m)} \rightarrow \mathbb{C}$ for the operator $L_k^{(m)}$ (2.12a) *with* the boundary convention (2.12b) via a suitable linear combinations of plane waves (corresponding to the same eigenvalue $E_k(\xi)$). This prompts us to look for eigenfunctions given by a linear combination of plane waves $\exp(i\langle \lambda, \xi_\sigma \rangle)$, $\sigma \in \mathcal{S}_n$ with coefficients such

that the boundary conditions in Proposition 2.1 are satisfied. Specifically, we will employ a \mathcal{S}_n -invariant (in ξ) Bethe Ansatz wave function of the form

$$\Psi_{\lambda}(\xi) = \frac{1}{\delta(\xi)} \sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} \mathcal{C}(\xi_{\sigma}) e^{i\langle \rho + \lambda, \xi_{\sigma} \rangle}, \quad \xi \in 2\pi \text{Int}(\mathbf{A}), \quad (3.1a)$$

where $\xi_{\sigma} := \sigma(\xi)$, $(-1)^{\sigma} := \det(\sigma) = (-1)^{\ell(\sigma)}$,

$$\delta(\xi) := \prod_{\alpha \in \mathbf{R}^+} (e^{i\langle \alpha, \xi \rangle / 2} - e^{-i\langle \alpha, \xi \rangle / 2}), \quad (3.1b)$$

and $\text{Int}(\mathbf{A}) = \{\xi \in \mathbf{E} \mid \langle \xi, \alpha_0 \rangle < 1, \langle \xi, \alpha_j \rangle > 0, j = 1, \dots, n-1\}$. (The condition that $\xi \in 2\pi \text{Int}(\mathbf{A})$ guarantees that the denominator $\delta(\xi)$ is nonzero.)

Proposition 3.1 (Bethe Wave Function). *The Bethe Ansatz wave function $\Psi_{\lambda}(\xi)$ (3.1a), (3.1b) satisfies the boundary reflection relations in Proposition 2.1 for $j = 1, \dots, n-1$ provided that*

$$\mathcal{C}(\xi) = \prod_{\alpha \in \mathbf{R}^+} (1 - t e^{-i\langle \alpha, \xi \rangle}) \quad (3.2)$$

(or a scalar multiple thereof).

Proof. Let $\lambda \in \mathcal{P}^{(m)}$ and $\nu \in \mathcal{S}_n(\omega_k)$ such that $\langle \lambda + \nu, \alpha_j \rangle = -1$ for some $j \in \{1, \dots, n-1\}$. Equating

$$\Psi_{\lambda+\nu}(\xi) = \frac{1}{\delta(\xi)} \sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} \mathcal{C}(\xi_{\sigma}) e^{i\langle \rho + \lambda + \nu, \xi_{\sigma} \rangle}$$

to

$$\begin{aligned} t \Psi_{r_j(\lambda+\nu)}(\xi) &= t \Psi_{\lambda+\nu+\alpha_j}(\xi) \\ &= \frac{t}{\delta(\xi)} \sum_{\sigma \in \mathcal{S}_n} (-1)^{\sigma} \mathcal{C}(\xi_{\sigma}) e^{i\langle \alpha_j, \xi_{\sigma} \rangle} e^{i\langle \rho + \lambda + \nu, \xi_{\sigma} \rangle} \end{aligned}$$

leads to the relation

$$\sum_{\sigma \in \mathcal{S}_n, \rho + \lambda + \nu} (-1)^{\sigma} \mathcal{C}(\xi_{\sigma\tau}) = t \sum_{\sigma \in \mathcal{S}_n, \rho + \lambda + \nu} (-1)^{\sigma} \mathcal{C}(\xi_{\sigma\tau}) e^{i\langle \alpha_j, \xi_{\sigma\tau} \rangle} \quad \forall \tau \in \mathcal{S}_n.$$

Because r_j stabilizes $\rho + \lambda + \nu$ (i.e. $r_j \in \mathcal{S}_n, \rho + \lambda + \nu$), the latter relation can be rewritten as

$$\begin{aligned} &\sum_{\substack{\sigma \in \mathcal{S}_n, \rho + \lambda + \nu \\ \sigma^{-1}(\alpha_j) \in \mathbf{R}^+}} (-1)^{\sigma} [\mathcal{C}(\xi_{\sigma\tau}) - \mathcal{C}(r_j(\xi_{\sigma\tau}))] \\ &= t \sum_{\substack{\sigma \in \mathcal{S}_n, \rho + \lambda + \nu \\ \sigma^{-1}(\alpha_j) \in \mathbf{R}^+}} (-1)^{\sigma} [\mathcal{C}(\xi_{\sigma\tau}) e^{i\langle \alpha_j, \xi_{\sigma\tau} \rangle} - \mathcal{C}(r_j(\xi_{\sigma\tau})) e^{-i\langle \alpha_j, \xi_{\sigma\tau} \rangle}]. \end{aligned}$$

By varying λ and ν it is seen that this relation implies that

$$\mathcal{C}(\xi) - \mathcal{C}(r_j(\xi)) = t [\mathcal{C}(\xi) e^{i\langle \alpha_j, \xi \rangle} - \mathcal{C}(r_j(\xi)) e^{-i\langle \alpha_j, \xi \rangle}]$$

(as an identity in ξ) for all reflections r_j , $j = 1, \dots, n-1$, or equivalently (assuming $\mathcal{C}(\xi)$ is nontrivial in the sense that it does not vanish identically)

$$\frac{\mathcal{C}(\xi)}{\mathcal{C}(r_j(\xi))} = \frac{1 - t e^{-i\langle \alpha_j, \xi \rangle}}{1 - t e^{i\langle \alpha_j, \xi \rangle}}, \quad j = 1, \dots, n-1.$$

Hence $\mathcal{C}(\xi)$ must be of the form

$$\mathcal{C}(\xi) = c_0(\xi) \prod_{\alpha \in \mathbf{R}^+} (1 - t e^{-i\langle \alpha, \xi \rangle}),$$

where $c_0(\xi)$ denotes an arbitrary \mathcal{S}_n -invariant overall factor (i.e. $c_0(\xi_\sigma) = c_0(\xi)$, $\forall \sigma \in \mathcal{S}_n$). \square

3.2. Bethe Equations. By pulling the overall factor $\delta(\xi)$ inside the sum and exploiting the anti-invariance $\delta(\xi_\sigma) = (-1)^\sigma \delta(\xi)$, the Bethe wave function $\Psi_\lambda(\xi)$ (3.1a), (3.1b), with coefficients $\mathcal{C}(\xi)$ taken from Eq. (3.2), passes over to

$$\Psi_\lambda(\xi) = \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{\alpha \in \mathbf{R}^+} \frac{1 - t e^{-i\langle \alpha, \xi_\sigma \rangle}}{1 - e^{-i\langle \alpha, \xi_\sigma \rangle}} \right) e^{i\langle \lambda, \xi_\sigma \rangle}. \quad (3.3)$$

From this expression it is clear that—for λ fixed—the Bethe wave function amounts to a Hall-Littlewood polynomial in the spectral parameter ξ [M2, M3]. We will now derive conditions on the spectral parameter such that the Bethe wave function satisfies the boundary reflection relations in Eqs. (2.13a), (2.13b) for $j = 0$.

Proposition 3.2 (Bethe System). *The Bethe Ansatz wave function $\Psi_\lambda(\xi)$ (3.3) satisfies the boundary reflection relations in Proposition 2.1 for $j = 0$ if the spectral parameter $\xi \in 2\pi \text{Int}(\mathbf{A})$ solves the algebraic system*

$$e^{im\langle \beta, \xi \rangle} = \left(\frac{1 - t e^{i\langle \beta, \xi \rangle}}{e^{i\langle \beta, \xi \rangle} - t} \right)^2 \prod_{\substack{\alpha \in \mathbf{R} \\ \langle \alpha, \beta \rangle = 1}} \frac{1 - t e^{i\langle \alpha, \xi \rangle}}{e^{i\langle \alpha, \xi \rangle} - t}, \quad \forall \beta \in \mathbf{R}. \quad (3.4)$$

Proof. Let us write $\hat{\mathcal{C}}(\xi) := \prod_{\alpha \in \mathbf{R}^+} \frac{1 - t e^{-i\langle \alpha, \xi \rangle}}{1 - e^{-i\langle \alpha, \xi \rangle}}$ and let $\lambda \in \mathcal{P}^{(m)}$ and $\nu \in \mathcal{S}_n(\omega_k)$ such that $\langle \lambda + \nu, \alpha_0 \rangle = m + 1$. Equating

$$\Psi_{\lambda+\nu}(\xi) = \sum_{\sigma \in \mathcal{S}_n} \hat{\mathcal{C}}(\xi_\sigma) e^{i\langle \lambda+\nu, \xi_\sigma \rangle}$$

to

$$t \Psi_{r_0^{(m)}(\lambda+\nu)}(\xi) = t \sum_{\sigma \in \mathcal{S}_n} \hat{\mathcal{C}}(\xi_\sigma) e^{i\langle r_0^{(m)}(\lambda+\nu), \xi_\sigma \rangle}$$

yields the relation

$$\sum_{\sigma \in \mathcal{S}_n} (1 - t e^{-i\langle \alpha_0, \xi_\sigma \rangle}) \hat{\mathcal{C}}(\xi_\sigma) e^{i\langle \lambda+\nu, \xi_\sigma \rangle} = 0,$$

or equivalently

$$\sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma^{-1}(\alpha_0) \in \mathbf{R}^+}} \left((1 - t e^{-i\langle \alpha_0, \xi_\sigma \rangle}) \hat{\mathcal{C}}(\xi_\sigma) e^{i\langle \lambda+\nu, \xi_\sigma \rangle} + (1 - t e^{-i\langle \alpha_0, r_0^{(0)}(\xi_\sigma) \rangle}) \hat{\mathcal{C}}(r_0^{(0)}(\xi_\sigma)) e^{i\langle \lambda+\nu, r_0^{(0)}(\xi_\sigma) \rangle} \right) = 0,$$

where $r_0^{(0)} \in \mathcal{S}_n$ denotes the orthogonal reflection $r_0^{(0)}(\mathbf{x}) = \mathbf{x} - \langle \mathbf{x}, \alpha_0 \rangle \alpha_0$. The latter equation translates to

$$\sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma^{-1}(\alpha_0) \in \mathbf{R}^+}} \left((1 - t e^{-i\langle \alpha_0, \xi_\sigma \rangle}) \hat{\mathcal{C}}(\xi_\sigma) + (1 - t e^{i\langle \alpha_0, \xi_\sigma \rangle}) \hat{\mathcal{C}}(r_0^{(0)}(\xi_\sigma)) e^{-(m+1)i\langle \alpha_0, \xi_\sigma \rangle} \right) e^{i\langle \lambda+\nu, \xi_\sigma \rangle} = 0,$$

which is satisfied if

$$(1 - t e^{-i\langle \alpha_0, \xi_\sigma \rangle}) \hat{\mathcal{C}}(\xi_\sigma) + (1 - t e^{i\langle \alpha_0, \xi_\sigma \rangle}) \hat{\mathcal{C}}(r_0^{(0)}(\xi_\sigma)) e^{-(m+1)i\langle \alpha_0, \xi_\sigma \rangle} = 0$$

for all $\sigma \in \mathcal{S}_n$, or equivalently (assuming $\hat{\mathcal{C}}(\xi_\sigma) \neq 0$)

$$e^{(m+1)i\langle \beta, \xi \rangle} = - \frac{\hat{\mathcal{C}}(r_0^{(0)}(\xi_\sigma))}{\hat{\mathcal{C}}(\xi_\sigma)} \frac{1 - t e^{i\langle \beta, \xi \rangle}}{1 - t e^{-i\langle \beta, \xi \rangle}}, \quad \forall \sigma \in \mathcal{S}_n,$$

where $\beta := \sigma^{-1}(\alpha_0)$. The proposition now follows upon inserting

$$\frac{\hat{\mathcal{C}}(r_0^{(0)}(\xi_\sigma))}{\hat{\mathcal{C}}(\xi_\sigma)} = - \prod_{\substack{\alpha \in \mathbf{R}^+ \\ \langle \alpha, \alpha_0 \rangle > 0}} \frac{1 - t e^{i\langle \alpha, \xi_\sigma \rangle}}{e^{i\langle \alpha, \xi_\sigma \rangle} - t} = - \prod_{\substack{\alpha \in \mathbf{R} \\ \langle \alpha, \beta \rangle > 0}} \frac{1 - t e^{i\langle \alpha, \xi \rangle}}{e^{i\langle \alpha, \xi \rangle} - t}.$$

□

4. SOLUTION OF THE BETHE EQUATIONS

In this section the Bethe System in Proposition 3.2 is solved using a variational technique due to C.N. Yang and C.P. Yang [YY, Ma, G4, KBI].

4.1. Solution. The following theorem provides (the existence of) a sequence $\xi_\mu \in 2\pi \text{Int}(\mathbf{A})$ of solutions to the Bethe system in Proposition 3.2 labeled by vectors (playing the role of quantum numbers) $\mu \in \mathcal{P}^{(m)}$.

Theorem 4.1 (Bethe Vectors). *For each $\mu \in \mathcal{P}^{(m)}$ there exists a (unique) Bethe vector $\xi_\mu \in 2\pi \text{Int}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \alpha_0 \rangle < 2\pi, \langle \mathbf{x}, \alpha_j \rangle > 0, j = 1, \dots, n-1\}$ such that ξ_μ satisfies the system in Eq. (3.4). Moreover, these Bethe vectors have the following properties:*

- (i) $\xi_{\mu'} = \xi_\mu$ if and only if $\mu' = \mu$,
- (ii) ξ_μ depends smoothly on the boundary parameter $t \in (-1, 1)$,
- (iii) $\xi_\mu = \frac{2\pi}{n+m}(\rho + \mu)$ for $t = 0$.

4.2. Proof. In standard coordinates the Bethe system of Proposition 3.2 reads

$$e^{im(\xi_j - \xi_k)} = \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \frac{1 - t e^{i(\xi_j - \xi_\ell)}}{e^{i(\xi_j - \xi_\ell)} - t} \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq k}} \frac{1 - t e^{i(\xi_\ell - \xi_k)}}{e^{i(\xi_\ell - \xi_k)} - t}, \quad (4.1)$$

for $1 \leq j \neq k \leq n$. This overdetermined system of $n(n-1)$ equations in the variables ξ_1, \dots, ξ_n is equivalent to the system of n equations

$$e^{im\xi_j} = c \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \frac{1 - t e^{i(\xi_j - \xi_\ell)}}{e^{i(\xi_j - \xi_\ell)} - t}, \quad j = 1, \dots, n, \quad (4.2)$$

where $c \neq 0$ denotes an overall constant factor that we can scale to 1 by means of the translation $\xi_j \rightarrow \xi_j - im^{-1} \log c$, $j = 1, \dots, n$. Picking thus $c = 1$ and taking the logarithm of both sides recasts Eq. (4.2) in the additive form

$$m\xi_j + \sum_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \theta(\xi_j - \xi_\ell) = 2\pi m_j, \quad j = 1, \dots, n, \quad (4.3)$$

where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and

$$\theta(x) := (1 - t^2) \int_0^x (1 - 2t \cos(x) + t^2)^{-1} dx \quad (4.4a)$$

$$= 2 \arctan \left(\frac{1+t}{1-t} \tan\left(\frac{x}{2}\right) \right) \quad (4.4b)$$

$$= i \log \left(\frac{1 - te^{ix}}{e^{ix} - t} \right). \quad (4.4c)$$

Here the branches of the arctangent function and those of the logarithmic function are to be chosen in such a way that (i) $\theta(x)$ (4.4b), (4.4c) is quasi-periodic: $\theta(x + 2\pi) = \theta(x) + 2\pi$, and (ii) $\theta(x)$ (4.4b), (4.4c) varies from $-\pi$ to π as x varies from $-\pi$ to π (which corresponds to the principal branch). We notice that this choice of the branches ensures that $\theta(x)$ (4.4b), (4.4c) is smooth on the whole real axis and strictly monotonously increasing.

Lemma 4.2. *For each n -tuple $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, there exists a unique vector $\boldsymbol{\xi}(\mathbf{m}) = (\xi_1(\mathbf{m}), \dots, \xi_n(\mathbf{m}))$ solving the system in Eq. (4.3) (with $\theta(x)$ of the form in Eqs. (4.4a)–(4.4c)). Furthermore, this solution $\boldsymbol{\xi}(\mathbf{m})$ depends smoothly on the boundary parameter $t \in (-1, 1)$.*

Proof. Let

$$V(\xi_1, \dots, \xi_n) := \frac{m}{2} \sum_{j=1}^n \xi_j^2 + \frac{1}{2} \sum_{j,k=1}^n \Theta(\xi_j - \xi_k) - 2\pi \sum_{j=1}^n m_j \xi_j, \quad (4.5)$$

where $\Theta(x) := \int_0^x \theta(x) dx$. Clearly the solution(s) of the system in Eq. (4.3) coincide with the critical point(s) of the (smooth) function $V(\xi_1, \dots, \xi_n)$. The Hesse matrix of V is given by

$$H_{j,k} = \frac{\partial^2 V}{\partial \xi_j \partial \xi_k} = \left(m + \sum_{\ell=1}^n \theta'(\xi_j - \xi_\ell) \right) \delta_{j,k} - \theta'(\xi_j - \xi_k), \quad 1 \leq j, k \leq n,$$

where $\theta'(x) = (1 - t^2)(1 - 2t \cos(x) + t^2)^{-1} > 0$. It is readily seen that this Hesse matrix is positive definite:

$$\sum_{j,k=1}^n H_{j,k} x_j x_k = m \sum_{j=1}^n x_j^2 + \frac{1}{2} \sum_{j,k=1}^n \theta'(\xi_j - \xi_k) (x_j - x_k)^2 \geq m \sum_{j=1}^n x_j^2 > 0$$

(for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$). The function $V(\xi_1, \dots, \xi_n)$ is thus strictly convex, i.e., it admits at most *one* critical point: a global minimum. That such global minimum $\boldsymbol{\xi}(\mathbf{m})$ indeed exists in our case is immediate from the observation that $V(\xi_1, \dots, \xi_n) \rightarrow +\infty$ when $\|\boldsymbol{\xi}\| \rightarrow \infty$. (Notice in this connection that $\Theta(x) \rightarrow +\infty$ for $x \rightarrow \pm\infty$.) We thus conclude that the system in Eq. (4.3) has a unique solution $\boldsymbol{\xi}(\mathbf{m})$ (given by the global minimum of V). It remains to check that the position of this global minimum depends smoothly on the boundary parameter t . To this end we notice that the integrand of $\theta(x)$ (4.4a) (which, incidentally, coincides with the generating function for the Chebyshev polynomials) is analytic in t for $|t| < 1$, and thus so are the function $V(\xi_1, \dots, \xi_n)$ and the system of equations for the global minimum in Eq. (4.3). The smoothness in ξ_1, \dots, ξ_n and t —combined with the fact that the Hessian $\det(H_{j,k})$ is positive (i.e. nonvanishing)—now guarantees that the solution $\boldsymbol{\xi}(\mathbf{m})$ to the latter system must be smooth in $t \in (-1, 1)$ by the implicit function theorem. \square

The following lemma shows that the ordering between the components of the solution $\xi(\mathbf{m})$ coincides with the ordering of the components of the labeling vector \mathbf{m} .

Lemma 4.3. *Let $\mathbf{m} \in \mathbb{Z}^n$ and let $\xi(\mathbf{m})$ be the associated solution of Eq. (4.3) detailed in Lemma 4.2. Then for $m_j \geq m_k$ the following inequalities hold*

$$\frac{2\pi(m_j - m_k)}{m + n\kappa_-(t)} \leq \xi_j(\mathbf{m}) - \xi_k(\mathbf{m}) \leq \frac{2\pi(m_j - m_k)}{m + n\kappa_+(t)}, \quad (4.6a)$$

where $\kappa_{\pm}(t) := \frac{1-t^2}{(1\pm|t|)^2} > 0$. So, one has in particular that

$$m_j > m_k \implies \xi_j(\mathbf{m}) > \xi_k(\mathbf{m}) \quad \text{and} \quad m_j = m_k \implies \xi_j(\mathbf{m}) = \xi_k(\mathbf{m}). \quad (4.6b)$$

Proof. Let \mathbf{m} in \mathbb{Z}^n with $m_j \geq m_k$. Subtracting the k^{th} equation from the j^{th} equation of the system in Eq. (4.3) yields that

$$m(\xi_j - \xi_k) + \sum_{\ell=1}^n (\theta(\xi_j - \xi_\ell) - \theta(\xi_k - \xi_\ell)) = 2\pi(m_j - m_k). \quad (4.7)$$

Since the r.h.s. of this identity is nonnegative and $\theta(x)$ is strictly monotonously increasing, it follows that $\xi_j(\mathbf{m}) \geq \xi_k(\mathbf{m})$. Furthermore, from the formula $\theta(x) - \theta(y) = (1-t^2) \int_y^x (1-2t \cos(x) + t^2)^{-1} dx$ (cf. Eq. (4.4a)) it is immediate that

$$\kappa_+(t)(x-y) \leq \theta(x) - \theta(y) \leq \kappa_-(t)(x-y) \quad \text{for } x \geq y.$$

Application of this upper and lower bound so as to estimate the terms in the sums on the l.h.s. of Eq. (4.7), now gives rise to the inequalities

$$(m + n\kappa_+(t))(\xi_j(\mathbf{m}) - \xi_k(\mathbf{m})) \leq 2\pi(m_j - m_k) \leq (m + n\kappa_-(t))(\xi_j(\mathbf{m}) - \xi_k(\mathbf{m})),$$

which completes the proof of Eq. (4.6a) (and thus also that of Eq. (4.6b)). \square

The next lemma improves the upper bound on the distance between the $\xi_j(\mathbf{m})$ and $\xi_k(\mathbf{m})$ stemming from Lemma 4.3 in the situation that the distance between m_j and m_k is smaller than $n + m$.

Lemma 4.4. *Let $\mathbf{m} \in \mathbb{Z}^n$ such that $m_j - m_k < n + m$ and let $\xi(\mathbf{m})$ be the associated solution of Eq. (4.3) detailed in Lemma 4.2. Then*

$$\xi_j(\mathbf{m}) - \xi_k(\mathbf{m}) < 2\pi. \quad (4.8)$$

Proof. Subtracting the k^{th} equation from the j^{th} equation of the system in Eq. (4.3) leads—upon recalling that $\theta(x)$ is odd—to (cf. Eq. (4.7))

$$m(\xi_j - \xi_k) + \sum_{\ell=1}^n (\theta(\xi_j - \xi_\ell) + \theta(\xi_\ell - \xi_k)) = 2\pi(m_j - m_k). \quad (4.9)$$

If $\xi_j - \xi_k \geq 2\pi$, then the average of $\xi_j - \xi_\ell$ and $\xi_\ell - \xi_k$ is $\geq \pi$. Hence $\theta(\xi_j - \xi_\ell) + \theta(\xi_\ell - \xi_k) \geq 2\pi$, in view of the fact that $\theta(x)$ is strictly monotonously increasing and $\theta(\pi + x) + \theta(\pi - x) = 2\pi$. Plugging this estimate in Eq. (4.9) reveals that $\xi_j(\mathbf{m}) - \xi_k(\mathbf{m}) \geq 2\pi$ implies that $2\pi(m_j - m_k) \geq m(\xi_j(\mathbf{m}) - \xi_k(\mathbf{m})) + 2\pi n \geq 2\pi(m + n)$, which completes the proof. \square

We will now piece the results of Lemmas 4.2–4.4 together, so as to arrive at a proof for Theorem 4.1.

Let $\mathbf{m} \in \mathbb{Z}^n$ such that

$$m_1 > m_2 > \cdots > m_n \quad \text{and} \quad m_1 - m_n < m + n, \quad (4.10a)$$

and let $\xi(\mathbf{m})$ be the associated solution of Eq. (4.3) detailed in Lemma 4.2. It follows from Lemmas 4.3 and 4.4 that

$$\xi_1(\mathbf{m}) > \xi_2(\mathbf{m}) > \cdots > \xi_n(\mathbf{m}) \quad \text{and} \quad \xi_1(\mathbf{m}) - \xi_n(\mathbf{m}) < 2\pi. \quad (4.10b)$$

Let us define

$$\boldsymbol{\mu} := \mathbf{m} - \frac{1}{n} \langle \mathbf{m}, \mathbf{e} \rangle \mathbf{e} - \boldsymbol{\rho}, \quad \boldsymbol{\xi}_\mu := \boldsymbol{\xi}(\mathbf{m}) - \frac{1}{n} \langle \boldsymbol{\xi}(\mathbf{m}), \mathbf{e} \rangle \mathbf{e}, \quad (4.11)$$

where (recall) $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_n$ and $\boldsymbol{\rho} = \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_{n-1}$. In other words, $\boldsymbol{\mu}$ is the orthogonal projection of \mathbf{m} onto the center-of-mass hyperplane \mathbf{E} (2.1) translated by $-\boldsymbol{\rho}$ and $\boldsymbol{\xi}_\mu$ is the orthogonal projection of $\boldsymbol{\xi}(\mathbf{m})$ onto \mathbf{E} . The inequalities in Eqs. (4.10a) and (4.10b) ensure that $\boldsymbol{\mu} \in \mathcal{P}^{(m)}$ (2.11) and that $\boldsymbol{\xi}_\mu \in 2\pi \text{Int}(\mathbf{A})$ (cf. Eq. (2.4)), respectively. It is furthermore clear that by varying \mathbf{m} we can reach any lattice point $\boldsymbol{\mu} \in \mathcal{P}^{(m)}$. Indeed, for $\boldsymbol{\mu} = k_1 \boldsymbol{\omega}_1 + \cdots + k_{n-1} \boldsymbol{\omega}_{n-1}$ with $k_j \in \mathbb{Z}_{\geq 0}$ and $k_1 + \cdots + k_{n-1} \leq m$ we may pick the components of \mathbf{m} equal to $m_j = \langle \boldsymbol{\mu} + \boldsymbol{\rho}, \boldsymbol{\alpha}_j + \cdots + \boldsymbol{\alpha}_{n-1} \rangle = k_j + \cdots + k_{n-1} + n - j$, $j = 1, \dots, n$. It is not difficult to check that the assignment $\boldsymbol{\mu} \rightarrow \boldsymbol{\xi}_\mu$ is indeed well-defined (i.e. $\boldsymbol{\mu} = \boldsymbol{\mu}' \Rightarrow \boldsymbol{\xi}_\mu = \boldsymbol{\xi}_{\mu'}$) and one-to-one (i.e. $\boldsymbol{\xi}_\mu = \boldsymbol{\xi}_{\mu'} \Rightarrow \boldsymbol{\mu} = \boldsymbol{\mu}'$). Indeed, one has that

$$\begin{aligned} \boldsymbol{\xi}_\mu = \boldsymbol{\xi}_{\mu'} &\iff \boldsymbol{\xi}(\mathbf{m}) - \boldsymbol{\xi}(\mathbf{m}') \in \mathbb{R}\mathbf{e} \\ &\stackrel{\text{Eq. (4.3)}}{\iff} \boldsymbol{\xi}(\mathbf{m}) - \boldsymbol{\xi}(\mathbf{m}') \in 2\pi\mathbb{Z}\mathbf{e} \\ &\stackrel{\text{Eq. (4.3)}}{\iff} \mathbf{m} - \mathbf{m}' \in \mathbb{Z}\mathbf{e} \\ &\iff \boldsymbol{\mu} = \boldsymbol{\mu}'. \end{aligned}$$

Since it is obvious that $\boldsymbol{\xi}_\mu$ inherits from $\boldsymbol{\xi}(\mathbf{m})$ the smooth dependence on the boundary parameter t and the property that its components solve the system in Eq. (4.2) (and thus the Bethe system in Eq. (4.1)), this proves Theorem 4.1 up to Property (ii). It remains to check Property (iii), which states that for $t = 0$ the Bethe vectors are given by $\boldsymbol{\xi}_\mu = \frac{2\pi}{n+m}(\boldsymbol{\rho} + \boldsymbol{\mu})$, $\boldsymbol{\mu} \in \mathcal{P}^{(m)}$. To this end we simply observe that Lemma 4.3 implies that for $t = 0$

$$\xi_j(\mathbf{m}) - \xi_k(\mathbf{m}) = \frac{2\pi}{n+m}(m_j - m_k),$$

whence the statement follows by varying \mathbf{m} subject to the constraints in Eq. (4.10a) and projecting onto the center-of-mass plane with the aid of Eq. (4.11).

5. DIAGONALIZATION

In this section we will combine the results of Sections 2–4 to arrive at an orthogonal basis for the Hilbert space $\mathcal{H}^{(m)} = \ell^2(\mathcal{P}^{(m)}, \Delta^{(m)})$, consisting of a complete set of joint eigenfunctions for the (commuting) Laplace operators $L_1^{(m)}, \dots, L_{n-1}^{(m)}$.

5.1. Spectrum and Eigenfunctions. The following theorem provides the eigenfunctions of our Laplace operators in terms of Hall-Littlewood polynomials specialized at the Bethe vectors $\boldsymbol{\xi}_\mu$, $\boldsymbol{\mu} \in \mathcal{P}^{(m)}$.

Theorem 5.1 (Spectrum and Eigenfunctions). *For special values of the spectral parameter, given by the Bethe vectors $\boldsymbol{\xi}_\mu$, $\boldsymbol{\mu} \in \mathcal{P}^{(m)}$ in Theorem 4.1, the Bethe*

wave function $\Psi_{\lambda}(\xi)$ (3.3) constitutes an eigenfunction of the Laplace operator $L_k^{(m)}$ (2.12a), (2.12b), i.e. for any $k \in \{1, \dots, n-1\}$

$$L_k^{(m)} \Psi(\xi_{\mu}) = E_k(\xi_{\mu}) \Psi(\xi_{\mu}), \quad (5.1a)$$

where the eigenvalue is of the form

$$E_k(\xi) = \sum_{\nu \in \mathcal{S}_n(\omega_k)} \exp(i\langle \nu, \xi \rangle) \quad (5.1b)$$

(and $\Psi(\xi_{\mu}) \neq 0$).

Proof. Clearly the Hall-Littlewood polynomials $\Psi_{\lambda}(\xi)$ (3.3) satisfy the identity $\sum_{\nu \in \mathcal{S}_n(\omega_k)} \Psi_{\lambda+\nu}(\xi) = E_k(\xi) \Psi_{\lambda}(\xi)$ (because all of the plane waves $\psi_{\lambda}(\xi_{\sigma}) = \exp(i\langle \lambda, \xi_{\sigma} \rangle)$, $\sigma \in \mathcal{S}_n$ do so). Moreover, since the specialized Hall-Littlewood polynomials $\Psi_{\lambda}(\xi_{\mu})$, $\mu \in \mathcal{P}^{(m)}$ also satisfy the boundary convention in Eq. (2.12b) in view of Propositions 2.1, 3.1, 3.2 and Theorem 4.1, the stated eigenvalue equation follows. It remains to check that $\Psi_{\lambda}(\xi_{\mu})$ does not vanish identically. For this purpose it is enough to observe that for $\lambda = \mathbf{0}$:

$$\Psi_{\mathbf{0}}(\xi) = \sum_{\sigma \in \mathcal{S}_n} \prod_{\alpha \in \mathbf{R}^+} \frac{1 - t e^{-i\langle \alpha, \xi_{\sigma} \rangle}}{1 - e^{-i\langle \alpha, \xi_{\sigma} \rangle}} \stackrel{(i)}{=} \sum_{\sigma \in \mathcal{S}_n} t^{\ell(\sigma)} \stackrel{(ii)}{=} \prod_{\alpha \in \mathbf{R}^+} \frac{1 - t^{1+\langle \rho, \alpha \rangle}}{1 - t^{\langle \rho, \alpha \rangle}},$$

where we have used (i) a rational function identity and (ii) a product formula for the Poincaré series of the permutation group that are both due to Macdonald [M1] (cf. Theorem (2.8) and Corollary (2.5), respectively). It is clear from the product formula on the r.h.s. that $\Psi_{\mathbf{0}}(\xi) > 0$ for $-1 < t < 1$, whence $\Psi_{\lambda}(\xi_{\mu})$ indeed constitutes a true (i.e. nonzero) eigenfunction in $\mathcal{H}^{(m)}$. \square

5.2. Orthogonality and Completeness. Theorem 5.1 provides as many eigenfunctions as the dimension of the Hilbert space (indeed, $\dim(\mathcal{H}^{(m)}) = \#\mathcal{P}^{(m)} = \binom{n+m-1}{m}$). The following theorem confirms our expectation that these eigenfunctions actually form an orthogonal basis for the Hilbert space in question. Alternatively, one may think of this theorem as describing a novel system of discrete (dual) orthogonality relations for the Hall-Littlewood polynomials.

Theorem 5.2 (Orthogonality and Completeness). *The Bethe wave functions*

$$\Psi(\xi_{\mu}), \quad \mu \in \mathcal{P}^{(m)} \quad (5.2a)$$

constitute an orthogonal basis of $\mathcal{H}^{(m)}$:

$$\forall \mu, \mu' \in \mathcal{P}^{(m)} : \quad \langle \Psi(\xi_{\mu}), \Psi(\xi_{\mu'}) \rangle^{(m)} = \begin{cases} 0 & \text{if } \mu \neq \mu', \\ > 0 & \text{if } \mu = \mu'. \end{cases} \quad (5.2b)$$

Proof. Since $L_k^{(m)}$ and $L_{n-k}^{(m)}$ are each others adjoints in $\mathcal{H}^{(m)}$ by Proposition 2.3, it is clear that $\langle L_k^{(m)} \Psi(\xi_{\mu}), \Psi(\xi_{\mu'}) \rangle^{(m)} = \langle \Psi(\xi_{\mu}), L_{n-k}^{(m)} \Psi(\xi_{\mu'}) \rangle^{(m)}$. By applying Theorem 5.1 and using the fact that $E_k(\xi) = \overline{E_{n-k}(\xi)}$, this equality is readily rewritten in the form

$$(E_k(\xi_{\mu}) - \overline{E_k(\xi_{\mu'})}) \langle \Psi(\xi_{\mu}), \Psi(\xi_{\mu'}) \rangle^{(m)} = 0. \quad (5.3)$$

Theorem 4.1 now guarantees that for $\mu \neq \mu'$ the associated Bethe vectors ξ_{μ} and $\xi_{\mu'}$ are distinct in $2\pi \text{Int}(\mathbf{A})$. Moreover, since the elementary symmetric polynomials

$E_1(\xi), \dots, E_{n-1}(\xi)$ separate the points of $2\pi\text{Int}(\mathbf{A})$ (as they generate the full algebra of trigonometric polynomials on $2\pi\mathbf{A}$ spanned by the \mathcal{S}_n -invariant Fourier basis $\sum_{\mu \in \mathcal{S}_n(\lambda)} \exp(i\langle \mu, \xi \rangle)$, $\lambda \in \mathcal{P}$), this implies that in this situation $E_k(\xi_\mu) \neq E_k(\xi_{\mu'})$ for a certain value of $k \in \{1, \dots, n-1\}$. We thus conclude from Eq. (5.3) that the inner product $\langle \Psi(\xi_\mu), \Psi(\xi_{\mu'}) \rangle^{(m)}$ must vanish if $\mu \neq \mu'$. Finally, for $\mu = \mu'$ the inner product yields the squared norm of the Bethe wave function $\Psi(\xi_\mu)$ in $\mathcal{H}^{(m)}$, which is positive as $\Psi(\xi_\mu) \neq 0$ by (the proof of) Theorem 5.1. \square

5.3. Integrability. From the previous results it is seen that our Laplace operators model a finite-dimensional quantum system that is integrable in the following sense.

Theorem 5.3 (Integrability). *The Laplacians $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ (2.12a), (2.12b) constitute $n-1$ ($= \dim(\mathbf{E})$) mutually commuting operators in the Hilbert space $\mathcal{H}^{(m)}$. Furthermore, any operator $L : \mathcal{H}^{(m)} \rightarrow \mathcal{H}^{(m)}$ that commutes with all of the Laplacians $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ lies in the polynomial algebra $\mathbb{C}[L_1^{(m)}, \dots, L_{n-1}^{(m)}]$.*

Proof. The commutativity of $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ is immediate from the fact that the operators are simultaneously diagonalized by the basis $\Psi(\xi_\mu)$, $\mu \in \mathcal{P}^{(m)}$ of $\mathcal{H}^{(m)}$ (cf. Theorems 5.1 and 5.2). The property that any operator $L : \mathcal{H}^{(m)} \rightarrow \mathcal{H}^{(m)}$ that commutes with $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ is necessarily algebraically dependent of $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ hinges on the fact that the eigenvalues $E_1(\xi_\mu), \dots, E_{n-1}(\xi_\mu)$ separate the elements of the eigenbasis $\Psi(\xi_\mu)$, $\mu \in \mathcal{P}^{(m)}$ (cf. also the proof of Theorem 5.2). Indeed, it is immediate from this that L is diagonalized by $\Psi(\xi_\mu)$, $\mu \in \mathcal{P}^{(m)}$. In other words, that there exist a function $E_L : \{\xi_\mu\}_{\mu \in \mathcal{P}^{(m)}} \rightarrow \mathbb{C}$ such that

$$L\Psi(\xi_\mu) = E_L(\xi_\mu)\Psi(\xi_\mu), \quad \forall \mu \in \mathcal{P}^{(m)}. \quad (5.4a)$$

Since the Bethe functions $\Psi(\xi_\mu)$, $\mu \in \mathcal{P}^{(m)}$ form an orthogonal basis of $\mathcal{H}^{(m)}$, we have (by transposition) that the Hall-Littlewood polynomials $\Psi_\lambda(\xi)$, $\lambda \in \mathcal{P}^{(m)}$ form a basis for the space of complex functions on the spectral set $\{\xi_\mu\}_{\mu \in \mathcal{P}^{(m)}}$ upon specialization. In particular, there exist (unique) complex coefficients c_λ , $\lambda \in \mathcal{P}^{(m)}$ such that

$$E_L(\xi_\mu) = \sum_{\lambda \in \mathcal{P}^{(m)}} c_\lambda \Psi_\lambda(\xi_\mu), \quad \forall \mu \in \mathcal{P}^{(m)}. \quad (5.4b)$$

Furthermore, from the well-known property that the elementary symmetric polynomials $E_1(\xi), \dots, E_{n-1}(\xi)$ (5.1b) generate the space of symmetric polynomials it is clear that there exist a polynomial $P_L \in \mathbb{C}[E_1, \dots, E_{n-1}]$ such that

$$\sum_{\lambda \in \mathcal{P}^{(m)}} c_\lambda \Psi_\lambda(\xi) = P_L(E_1(\xi), \dots, E_{n-1}(\xi)). \quad (5.4c)$$

It follows from Eqs. (5.4a)–(5.4c) and Theorem 5.1 that the operators L and $P_L(L_1^{(m)}, \dots, L_{n-1}^{(m)})$ coincide on the basis $\Psi(\xi_\mu)$, $\mu \in \mathcal{P}^{(m)}$. Hence, we conclude that $L = P_L(L_1^{(m)}, \dots, L_{n-1}^{(m)}) \in \mathbb{C}[L_1^{(m)}, \dots, L_{n-1}^{(m)}]$. \square

The Laplace operators $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ are not self-adjoint in general in view of Proposition 2.3. As a consequence, the spectrum in Theorem 5.1 is generally complex-valued. Within the commuting algebra $\mathbb{C}[L_1^{(m)}, \dots, L_{n-1}^{(m)}]$ there exist however many operators that are self-adjoint. For example, the alternative generators

$$L_{R,k}^{(m)} := \frac{1}{2}(L_k^{(m)} + L_{n-k}^{(m)}), \quad k \in \{1, \dots, [n/2]\}, \quad (5.5a)$$

$$L_{I,k}^{(m)} := \frac{1}{2i}(L_k^{(m)} - L_{n-k}^{(m)}), \quad k \in \{1, \dots, [(n-1)/2]\}, \quad (5.5b)$$

are self-adjoint and have real spectrum of the form

$$E_{R,k}(\xi_\mu) = \sum_{\nu \in \mathcal{S}_n(\omega_k)} \cos(\langle \nu, \xi_\mu \rangle), \quad \mu \in \mathcal{P}^{(m)}, \quad (5.6a)$$

$$E_{I,k}(\xi_\mu) = \sum_{\nu \in \mathcal{S}_n(\omega_k)} \sin(\langle \nu, \xi_\mu \rangle), \quad \mu \in \mathcal{P}^{(m)}, \quad (5.6b)$$

respectively. The real subalgebra $\mathbb{R}[L_{R,1}^{(m)}, \dots, L_{R,[n/2]}^{(m)}, L_{I,1}^{(m)}, \dots, L_{I,[(n-1)/2]}^{(m)}]$ consists of all operators $L : \mathcal{H}^{(m)} \rightarrow \mathcal{H}^{(m)}$ such that (i) L commutes with all of the Laplacians $L_1^{(m)}, \dots, L_{n-1}^{(m)}$ and (ii) L is self-adjoint.

One of the simplest positive operators in this real subalgebra is given by

$$H^{(m)} := n\text{Id} - L_{R,1}^{(m)}. \quad (5.7)$$

In standard coordinates the explicit action of this operator on an arbitrary wave function $\psi \in \mathcal{H}^{(m)}$ is of the form (cf. Proposition 2.2)

$$(H^{(m)}\psi)_\lambda = n\psi_\lambda - \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ \lambda + \nu_j \in \mathcal{P}^{(m)}}} V_{j,\lambda}^+ \psi_{\lambda + \nu_j} - \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ \lambda - \nu_j \in \mathcal{P}^{(m)}}} V_{j,\lambda}^- \psi_{\lambda - \nu_j}, \quad (5.8a)$$

where

$$V_{j,\lambda}^+ = \prod_{\substack{j < k \leq n \\ \lambda_k = \lambda_j}} \frac{1 - t^{1+k-j}}{1 - t^{k-j}} \prod_{\substack{1 \leq k < j \\ \lambda_k = \lambda_j + m}} \frac{1 - t^{1+n+k-j}}{1 - t^{n+k-j}}, \quad (5.8b)$$

$$V_{j,\lambda}^- = \prod_{\substack{1 \leq k < j \\ \lambda_k = \lambda_j}} \frac{1 - t^{1+j-k}}{1 - t^{j-k}} \prod_{\substack{j < k \leq n \\ \lambda_k = \lambda_j - m}} \frac{1 - t^{1+n+j-k}}{1 - t^{n+j-k}}, \quad (5.8c)$$

and $\nu_j = \mathbf{e}_j - (\mathbf{e}_1 + \dots + \mathbf{e}_n)/n$, $j = 1, \dots, n$ (so ν_1, \dots, ν_n consist of the orthogonal projection of the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ onto the center-of-mass plane \mathbf{E} (2.1)). The spectrum of $H^{(m)}$ is built of positive eigenvalues $E(\xi_\mu)$, $\mu \in \mathcal{P}^{(m)}$ with

$$E(\xi) = \sum_{j=1}^n (1 - \cos(\xi_j)). \quad (5.8d)$$

The operator $H^{(m)}$ (5.8a)–(5.8c) serves as the Hamiltonian of our lattice n -particle model. Below we will verify that in a continuum limit this lattice Hamiltonian tends formally to the Hamiltonian of the n -particle delta Bose gas on the circle.

6. CONTINUUM LIMIT

In this final section we first review the solution of the eigenvalue problem for the Laplacian in Eq. (1.2), with wave functions supported inside the alcove \mathbf{A} (1.3) subject to repulsive boundary conditions of the form in Eqs. (1.4a), (1.4b) (i.e. with $g > 0$). Our formulation amounts to the center-of-mass reduction of the seminal results due to Lieb and Liniger [LL] (Bethe wave functions), C.N. Yang and

C.P. Yang [YY] (Bethe vectors), and Dorlas [Do] (orthogonality and completeness). Next we will show how this solution of the eigenvalue problem for the Laplacian in the alcove can be recovered from the corresponding solution of our discrete lattice model via a continuum limit.

6.1. Eigenfunctions. In the notation of Section 2 the eigenvalue problem in Eqs. (1.2)–(1.4b) reads

$$-\Delta\psi = E\psi, \quad \mathbf{x} \in \mathbf{A}, \quad (6.1a)$$

with

$$(\langle \nabla_{\mathbf{x}} \psi, \boldsymbol{\alpha}_0 \rangle + g\psi)|_{\mathbf{x} \in \mathbf{E}_0} = 0, \quad (6.1b)$$

$$(\langle \nabla_{\mathbf{x}} \psi, \boldsymbol{\alpha}_j \rangle - g\psi)|_{\mathbf{x} \in \mathbf{E}_j} = 0, \quad j = 1, \dots, n-1 \quad (6.1c)$$

(where $\nabla_{\mathbf{x}}$ refers to the gradient). Let us define

$$\mathbf{C} := \{\boldsymbol{\xi} \in \mathbf{E} \mid \langle \boldsymbol{\xi}, \boldsymbol{\alpha}_j \rangle > 0, \quad j = 1, \dots, n-1\}, \quad (6.2a)$$

$$\mathcal{P}^{(\infty)} := \{k_1 \boldsymbol{\omega}_1 + \dots + k_{n-1} \boldsymbol{\omega}_{n-1} \mid k_1, \dots, k_{n-1} \in \mathbb{Z}_{\geq 0}\}. \quad (6.2b)$$

Theorem 6.1 (Bethe Wave Functions [LL]). *The Bethe wave function**

$$\Psi^{(\infty)}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{\boldsymbol{\alpha} \in \mathbf{R}^+} \frac{\langle \boldsymbol{\alpha}, \boldsymbol{\xi}_{\sigma} \rangle - ig}{\langle \boldsymbol{\alpha}, \boldsymbol{\xi}_{\sigma} \rangle} \right) e^{i\langle \mathbf{x}, \boldsymbol{\xi}_{\sigma} \rangle}, \quad (6.3a)$$

with the spectral parameter $\boldsymbol{\xi} \in \mathbf{C}$ (6.2a) solving the Bethe system

$$e^{i\langle \boldsymbol{\beta}, \boldsymbol{\xi} \rangle} = \left(\frac{ig + \langle \boldsymbol{\beta}, \boldsymbol{\xi} \rangle}{ig - \langle \boldsymbol{\beta}, \boldsymbol{\xi} \rangle} \right)^2 \prod_{\substack{\boldsymbol{\alpha} \in \mathbf{R} \\ \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = 1}} \frac{ig + \langle \boldsymbol{\alpha}, \boldsymbol{\xi} \rangle}{ig - \langle \boldsymbol{\alpha}, \boldsymbol{\xi} \rangle}, \quad \forall \boldsymbol{\beta} \in \mathbf{R}, \quad (6.3b)$$

constitutes a solution to the eigenvalue problem in Eqs. (6.1a)–(6.1c) corresponding to the eigenvalue $E = E^{(\infty)}(\boldsymbol{\xi}) := \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle$.

It is instructive to recall briefly the essence of the proof of Lieb and Liniger in the present notation. Firstly, it is clear that the linear combination of plane waves $\Psi^{(\infty)}(\mathbf{x}, \boldsymbol{\xi})$ constitutes an eigenfunction of $-\Delta$ with eigenvalue $E^{(\infty)}(\boldsymbol{\xi})$. It remains to check that the boundary conditions are also satisfied. The boundary condition

*This explicit form of the expressions for the coefficients of the Bethe wave function is due to Gaudin [G3, G4].

in Eq. (6.1c) is inferred by the following computation for $\mathbf{x} \in E_j$:

$$\begin{aligned}
& \langle \nabla_{\mathbf{x}} \Psi^{(\infty)}, \alpha_j \rangle \\
&= \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{\alpha \in \mathbf{R}^+} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle} \right) i \langle \alpha_j, \xi_\sigma \rangle e^{i \langle \mathbf{x}, \xi_\sigma \rangle} \\
&= \sum_{\sigma \in \mathcal{S}_n} (g + i \langle \alpha_j, \xi_\sigma \rangle) \left(\prod_{\substack{\alpha \in \mathbf{R}^+ \\ \alpha \neq \alpha_j}} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle} \right) e^{i \langle \mathbf{x}, \xi_\sigma \rangle} \\
&\stackrel{(i)}{=} g \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{\substack{\alpha \in \mathbf{R}^+ \\ \alpha \neq \alpha_j}} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle} \right) e^{i \langle \mathbf{x}, \xi_\sigma \rangle} \\
&\stackrel{(ii)}{=} g \sum_{\sigma \in \mathcal{S}_n} \left(1 - \frac{ig}{\langle \alpha_j, \xi_\sigma \rangle} \right) \left(\prod_{\substack{\alpha \in \mathbf{R}^+ \\ \alpha \neq \alpha_j}} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle} \right) e^{i \langle \mathbf{x}, \xi_\sigma \rangle} \\
&= g \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{\alpha \in \mathbf{R}^+} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle} \right) e^{i \langle \mathbf{x}, \xi_\sigma \rangle} \\
&= g \Psi^{(\infty)},
\end{aligned}$$

where in Steps (i) and (ii) one exploits that $\prod_{\substack{\alpha \in \mathbf{R}^+ \\ \alpha \neq \alpha_j}} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle}$ and $\langle \alpha_j, \xi_\sigma \rangle$ are symmetric and skew-symmetric, respectively, with respect to the action of r_j on ξ_σ , combined with the symmetry $\langle \mathbf{x}, r_j(\xi_\sigma) \rangle = \langle \mathbf{x}, \xi_\sigma \rangle$ (since $r_j(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in E_j$). Finally, the boundary condition in Eq. (6.1b) requires that for $\mathbf{x} \in E_0$

$$\sum_{\sigma \in \mathcal{S}_n} (g + i \langle \xi_\sigma, \alpha_0 \rangle) \left(\prod_{\alpha \in \mathbf{R}^+} \frac{\langle \alpha, \xi_\sigma \rangle - ig}{\langle \alpha, \xi_\sigma \rangle} \right) e^{i \langle \mathbf{x}, \xi_\sigma \rangle} = 0.$$

Manipulations similar to those in the proof of Proposition 3.2 reveal that this relation holds when the spectral parameter solves the Bethe system in Eq. (6.3b).

Theorem 6.2 (Bethe Vectors [YY]). *Let $g > 0$. For each $\mu \in \mathcal{P}^{(\infty)}$ (6.2b) there exists a (unique) Bethe vector $\xi_\mu \in \mathcal{C}$ (6.2a) such that ξ_μ satisfies the system in Eq. (6.3b). Moreover, these Bethe vectors have the following properties:*

- (i) $\xi_{\mu'} = \xi_\mu$ if and only if $\mu' = \mu$,
- (ii) ξ_μ depends smoothly on the boundary parameter $g > 0$,
- (iii) $\xi_\mu \rightarrow 2\pi(\rho + \mu)$ for $g \rightarrow +\infty$.

In standard coordinates the Bethe system in Eq. (6.3b) reads

$$e^{i(\xi_j - \xi_k)} = \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \frac{ig + \xi_j - \xi_\ell}{ig - \xi_j + \xi_\ell} \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq k}} \frac{ig + \xi_\ell - \xi_k}{ig - \xi_\ell + \xi_k}, \quad (6.4)$$

for $1 \leq j \neq k \leq n$, or equivalently (upon exploiting the translational invariance)

$$e^{i\xi_j} = \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \frac{ig + \xi_j - \xi_\ell}{ig - \xi_j + \xi_\ell}, \quad j = 1, \dots, n. \quad (6.5)$$

In the additive form the latter system becomes

$$\xi_j + \sum_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \theta^{(\infty)}(\xi_j - \xi_\ell) = 2\pi m_j, \quad j = 1, \dots, n, \quad (6.6)$$

with $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and

$$\theta^{(\infty)}(x) = 2g \int_0^x (x^2 + g^2)^{-1} dx \quad (6.7a)$$

$$= 2 \arctan\left(\frac{x}{g}\right) \quad (6.7b)$$

$$= i \log\left(\frac{ig + x}{ig - x}\right). \quad (6.7c)$$

It was shown in [YY] that for any $\mathbf{m} \in \mathbb{Z}^n$ the Bethe system in Eqs. (6.6)–(6.7c) has a unique solution $\boldsymbol{\xi}(\mathbf{m})$ given by the unique global minimum of the strictly convex function

$$V^{(\infty)}(\xi_1, \dots, \xi_n) := \frac{1}{2} \sum_{j=1}^n \xi_j^2 + \frac{1}{2} \sum_{j,k=1}^n \Theta^{(\infty)}(\xi_j - \xi_k) - 2\pi \sum_{j=1}^n m_j \xi_j, \quad (6.8)$$

with $\Theta^{(\infty)}(x) := \int_0^x \theta^{(\infty)}(x) dx$. By projecting the solutions $\boldsymbol{\xi}(\mathbf{m})$, corresponding to vectors $\mathbf{m} \in \mathbb{Z}^n$ with $m_1 > m_2 > \dots > m_n$, orthogonally onto the center-of-mass plane \mathbf{E} the statements of Theorem 6.2 readily follow (cf. also Section 4).

Theorem 6.3 (Orthogonality and Completeness [Do]). *The Bethe wave functions $\Psi^{(\infty)}(\mathbf{x}, \boldsymbol{\xi}_\mu)$, $\boldsymbol{\mu} \in \mathcal{P}^{(\infty)}$ form an orthogonal basis for the Hilbert space $\mathcal{H}^{(\infty)} := L^2(\mathbf{A}, d\mathbf{x})$ (with inner product $\langle \phi, \psi \rangle^{(\infty)} := \int_{\mathbf{A}} \phi(\mathbf{x}) \overline{\psi(\mathbf{x})} d\mathbf{x}$), i.e.*

$$\forall \boldsymbol{\mu}, \boldsymbol{\mu}' \in \mathcal{P}^{(\infty)} : \quad \langle \Psi^{(\infty)}(\boldsymbol{\xi}_\mu), \Psi^{(\infty)}(\boldsymbol{\xi}_{\mu'}) \rangle^{(\infty)} = \begin{cases} 0 & \text{if } \boldsymbol{\mu} \neq \boldsymbol{\mu}' \\ > 0 & \text{if } \boldsymbol{\mu} = \boldsymbol{\mu}' \end{cases} \quad (6.9a)$$

and

$$\langle \phi, \Psi^{(\infty)}(\boldsymbol{\xi}_\mu) \rangle = 0, \quad \forall \boldsymbol{\mu} \in \mathcal{P}^{(\infty)} \implies \phi = 0. \quad (6.9b)$$

Below we will infer that this center-of-mass reduction of Dorlas' orthogonality relations can be recovered via a continuum limit from the corresponding results pertaining to the discrete lattice model in Section 5. It was moreover shown by Dorlas that the orthogonality of the Bethe wave functions for the repulsive delta Boson gas implies their completeness [Do, Section 3]. In other words, the completeness in Theorem 6.3 follows from the orthogonality (upon a cosmetic adaptation of Dorlas' arguments to our center-of-mass situation).

6.2. Orthogonality. In order to perform the continuum limit let us from now on rescale the coupling parameter t putting

$$t = e^{-g/m}, \quad g > 0. \quad (6.10)$$

For any \mathbf{x} in (the closure of) \mathbf{C} (6.2a) we define an integral approximation $[\mathbf{x}] \in \mathcal{P}^{(\infty)}$ (6.2b) of the form

$$[\mathbf{x}] := [\langle \mathbf{x}, \boldsymbol{\alpha}_1 \rangle] \boldsymbol{\omega}_1 + \dots + [\langle \mathbf{x}, \boldsymbol{\alpha}_{n-1} \rangle] \boldsymbol{\omega}_{n-1}, \quad (6.11)$$

where $[x]$ denotes the integral part of $x \in \mathbb{R}_{\geq 0}$ obtained through truncation. With these notations we are in the position to embed the Hilbert space of lattice functions $\mathcal{H}^{(m)} = \ell^2(\mathcal{P}^{(m)}, \Delta^{(m)})$ into $L^2(\mathbf{C}, d\mathbf{x})$ by means of a linear injection $J^{(m)} : \mathcal{H}^{(m)} \rightarrow$

$L^2(\mathbf{C}, d\mathbf{x})$ that associates to a lattice function $\phi : \mathcal{P}^{(m)} \rightarrow \mathbb{C}$ a staircase function $J^{(m)}(\phi) : \mathbf{C} \rightarrow \mathbb{C}$ of the form

$$(J^{(m)}\phi)(\mathbf{x}) := \begin{cases} \sqrt{\Delta_{[m\mathbf{x}]}^{(m)}} \phi_{[m\mathbf{x}]} & \text{for } [m\mathbf{x}] \in \mathcal{P}^{(m)}, \\ 0 & \text{for } [m\mathbf{x}] \notin \mathcal{P}^{(m)}. \end{cases} \quad (6.12)$$

It is not difficult to see that the staircase function $J^{(m)}(\phi)$ has support on a bounded domain inside the dilated alcove $(1 + \frac{n}{m})\mathbf{A}$. This support shrinks towards (a subset of) \mathbf{A} for $m \rightarrow \infty$. It is also not difficult to deduce from this definition that $\forall \phi, \psi \in \mathcal{H}^{(m)}$

$$\int_{\mathbf{C}} (J^{(m)}\phi)(\mathbf{x}) \overline{(J^{(m)}\psi)(\mathbf{x})} d\mathbf{x} = c_{n,m} \sum_{\lambda \in \mathcal{P}^{(m)}} \phi_{\lambda} \overline{\psi_{\lambda}} \Delta_{\lambda}^{(m)}, \quad (6.13)$$

where $c_{n,m} = \text{Vol}(\omega_1, \dots, \omega_{n-1})/m^{n-1} = 1/(m^{n-1}\sqrt{n})$. Let $\Psi^{(m)}(\mathbf{x}, \boldsymbol{\xi})$ be the staircase embedding of the Hall-Littlewood polynomial $\Psi_{\lambda}(\boldsymbol{\xi})$ (3.3)

$$\begin{aligned} \Psi^{(m)}(\mathbf{x}, \boldsymbol{\xi}) &:= (J^{(m)}\Psi(\boldsymbol{\xi}))(\mathbf{x}) \\ &= \sqrt{\Delta_{[m\mathbf{x}]}^{(m)}} \Psi_{[m\mathbf{x}]}(\boldsymbol{\xi}), \\ &= \sqrt{\Delta_{[m\mathbf{x}]}^{(m)}} \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{\alpha \in \mathbf{R}^+} \frac{1 - e^{-g/m} e^{-i\langle \alpha, \boldsymbol{\xi}_{\sigma} \rangle}}{1 - e^{-i\langle \alpha, \boldsymbol{\xi}_{\sigma} \rangle}} \right) e^{i\langle [m\mathbf{x}], \boldsymbol{\xi}_{\sigma} \rangle}. \end{aligned} \quad (6.14)$$

The following lemma states that, for $m \rightarrow \infty$, the rescaled staircase function $\Psi^{(m)}(\mathbf{x}, \frac{1}{m}\boldsymbol{\xi})$ (6.14) converges pointwise to the Lieb-Liniger Bethe wave function $\Psi^{(\infty)}(\mathbf{x}, \boldsymbol{\xi})$ (6.3a) when \mathbf{x} lies in the interior of \mathbf{A} and to zero when \mathbf{x} lies outside \mathbf{A} .

Lemma 6.4. *For any $\boldsymbol{\xi} \in \mathbf{C}$, one has that*

$$\lim_{m \rightarrow \infty} \Psi^{(m)}(\mathbf{x}, \frac{1}{m}\boldsymbol{\xi}) = \begin{cases} \Psi^{(\infty)}(\mathbf{x}, \boldsymbol{\xi}) & \text{if } \mathbf{x} \in \text{Int}(\mathbf{A}), \\ 0 & \text{if } \mathbf{x} \in \mathbf{C} \setminus \mathbf{A}. \end{cases} \quad (6.15)$$

Proof. The lemma readily follows from the explicit expression of the staircase wave function on the third line of Eq. (6.14), together with the observation that $\lim_{m \rightarrow \infty} \Delta_{[m\mathbf{x}]}^{(m)} = 1$ if $\mathbf{x} \in \text{Int}(\mathbf{A})$ and $\lim_{m \rightarrow \infty} \Delta_{[m\mathbf{x}]}^{(m)} = 0$ if $\mathbf{x} \in \mathbf{C} \setminus \mathbf{A}$, and the fact that $\lim_{m \rightarrow \infty} \frac{1}{m}[m\mathbf{x}] = \mathbf{x}$. \square

Let us fix a $\boldsymbol{\mu} \in \mathcal{P}^{(\infty)}$ and pick m sufficiently large so as ensure that $\boldsymbol{\mu} \in \mathcal{P}^{(m)}$. We denote by $\boldsymbol{\xi}_{\boldsymbol{\mu}}^{(m)}$ and $\boldsymbol{\xi}_{\boldsymbol{\mu}}^{(\infty)}$ the associated Bethe vectors detailed in Theorem 4.1 and Theorem 6.2, respectively.

Lemma 6.5. *For any $\boldsymbol{\mu} \in \mathcal{P}^{(\infty)}$, one has that*

$$\lim_{m \rightarrow \infty} m \boldsymbol{\xi}_{\boldsymbol{\mu}}^{(m)} = \boldsymbol{\xi}_{\boldsymbol{\mu}}^{(\infty)}. \quad (6.16)$$

Proof. Let $m_j := \langle \boldsymbol{\mu}, \boldsymbol{\alpha}_j \rangle + \dots + \langle \boldsymbol{\mu}, \boldsymbol{\alpha}_{n-1} \rangle + n - j$, $j = 1, \dots, n$. Then $\boldsymbol{\xi}_{\boldsymbol{\mu}}^{(m)}$ and $\boldsymbol{\xi}_{\boldsymbol{\mu}}^{(\infty)}$ correspond to (the projections onto the center-of-mass plane of) the (unique) global minima of $V(\xi_1, \dots, \xi_n)$ (4.5) and $V^{(\infty)}(\xi_1, \dots, \xi_n)$ (6.8), respectively. The rescaled Bethe vector $m \boldsymbol{\xi}_{\boldsymbol{\mu}}^{(m)}$ thus corresponds to the global minimum of the function $V^{(m)}(\xi_1, \dots, \xi_n) := mV(\xi_1/m, \dots, \xi_n/m)$. The lemma now follows from the observation that for $m \rightarrow \infty$ the strictly convex function $V^{(m)}(\xi_1, \dots, \xi_n)$ tends

to $V^{(\infty)}(\xi_1, \dots, \xi_n)$ uniformly on compacts (which implies in particular that the global minimum of the $V^{(m)}$ converges to the global minimum of $V^{(\infty)}$). \square

The proof of the orthogonality in Theorem 6.3 now hinges on the following proposition.

Proposition 6.6. *For all $\mu, \mu' \in \mathcal{P}^{(\infty)}$, one has that*

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_C \Psi^{(m)}(\mathbf{x}, \xi_\mu^{(m)}) \overline{\Psi^{(m)}(\mathbf{x}, \xi_{\mu'}^{(m)})} d\mathbf{x} \\ = \int_A \Psi^{(\infty)}(\mathbf{x}, \xi_\mu^{(\infty)}) \overline{\Psi^{(\infty)}(\mathbf{x}, \xi_{\mu'}^{(\infty)})} d\mathbf{x}. \end{aligned} \quad (6.17)$$

Proof. It is clear from (the proof of) Lemma 6.4 and from Lemma 6.5 that the integrand and support of the integral on the l.h.s. converges pointwise to the integrand and support of the integral on the r.h.s. To see that the integrals themselves converge accordingly we write

$$\begin{aligned} \int_C \Psi^{(m)}(\mathbf{x}, \xi) \overline{\Psi^{(m)}(\mathbf{x}, \xi')} d\mathbf{x} \\ = \sum_{\sigma, \sigma' \in \mathcal{S}_n} \hat{\mathcal{C}}(\xi_\sigma) \hat{\mathcal{C}}(-\xi'_{\sigma'}) \int_{(1+\frac{n}{m})A} e^{i\langle [m\mathbf{x}], \xi_\sigma - \xi'_{\sigma'} \rangle} \Delta_{[m\mathbf{x}]}^{(m)} d\mathbf{x}, \end{aligned}$$

where $\hat{\mathcal{C}}(\xi) = \prod_{\alpha \in \mathbf{R}^+} \frac{1 - e^{-g/m} e^{-i\langle \alpha, \xi \rangle}}{1 - e^{-i\langle \alpha, \xi \rangle}}$. After substituting $\xi := \xi_\mu^{(m)}$ and $\xi' := \xi_{\mu'}^{(m)}$ the proposition follows for $m \rightarrow \infty$ upon invoking Lemma 6.5 and the dominated convergence theorem of Lebesgue. Indeed, one has that

$$e^{i\frac{1}{m} \langle [m\mathbf{x}], m\sigma(\xi_\mu^{(m)}) - m\sigma'(\xi_{\mu'}^{(m)}) \rangle} \longrightarrow e^{i\langle \mathbf{x}, \sigma(\xi_\mu^{(\infty)}) - \sigma'(\xi_{\mu'}^{(\infty)}) \rangle}$$

and

$$\Delta_{[m\mathbf{x}]}^{(m)} \longrightarrow \begin{cases} 1 & \text{if } \mathbf{x} \in \text{Int}(A) \\ 0 & \text{if } \mathbf{x} \in C \setminus A \end{cases}$$

pointwise for $m \rightarrow \infty$, and that $|e^{i\frac{1}{m} \langle [m\mathbf{x}], m\sigma(\xi_\mu^{(m)}) - m\sigma'(\xi_{\mu'}^{(m)}) \rangle}| = 1$, $|\Delta_{[m\mathbf{x}]}^{(m)}| \leq 1$. \square

Proposition 6.6 can be rephrased as

$$\begin{aligned} \langle \Psi^{(\infty)}(\xi_\mu^{(\infty)}), \Psi^{(\infty)}(\xi_{\mu'}^{(\infty)}) \rangle^{(\infty)} = \\ \lim_{m \rightarrow \infty} \int_C (J^{(m)} \Psi(\xi_\mu^{(m)}))(\mathbf{x}) \overline{(J^{(m)} \Psi(\xi_{\mu'}^{(m)}))(\mathbf{x})} d\mathbf{x}. \end{aligned}$$

The r.h.s. of this limiting relation vanishes when $\mu \neq \mu'$ in view of Eq. (6.13) and Theorem 5.2, whence the orthogonality in Theorem 6.3 follows.

6.3. Hamiltonian. We will now wrap up by verifying briefly that formally the Hamiltonian $H^{(m)}$ (5.8a), (5.8c) converges in the continuum limit to the Hamiltonian of the repulsive delta Bose gas on the circle. It is quite plausible that with a somewhat more in-depth analysis in the spirit of Ref. [R] one would be able to show that this convergence of the Hamiltonian is in fact in the strong resolvent sense, but we will not attempt to do so here.

Let $H^{(\infty)}$ be the self-adjoint extension in $\mathcal{H}^{(\infty)} = L^2(A, d\mathbf{x})$ of the Laplace operator $-\Delta$ with boundary conditions of the form in Eqs. (6.1b), (6.1c), and let

$H^{(m)}$ be the following rescaled staircase embedding of the operator $H^{(m)}$ (5.8a)–(5.8c) in $L^2(C, d\mathbf{x})$:

$$H^{(m)} = 2m^2 J^{(m)} H^{(m)} (J^{(m)})^{-1} \Pi^{(m)}, \quad (6.18)$$

where $\Pi^{(m)} : L^2(C, d\mathbf{x}) \rightarrow L^2(C, d\mathbf{x})$ denotes the orthogonal projection onto the finite-dimensional subspace of staircase functions $J^{(m)}(\mathcal{H}^{(m)}) \subset L^2(C, d\mathbf{x})$. It is clear that

$$H^{(\infty)} \Psi^{(\infty)}(\xi_\mu^{(\infty)}) = E^{(\infty)}(\xi_\mu^{(\infty)}) \Psi^{(\infty)}(\xi_\mu^{(\infty)}), \quad (6.19a)$$

with $E^{(\infty)}(\xi) = \langle \xi, \xi \rangle$, and that

$$H^{(m)} \Psi^{(m)}(\xi_\mu^{(m)}) = E^{(m)}(\xi_\mu^{(m)}) \Psi^{(m)}(\xi_\mu^{(m)}), \quad (6.19b)$$

where $E^{(m)}(\xi) := 2m^2 E(\xi)$ with $E(\xi)$ given by Eq. (5.8d). From Lemmas 6.4 and 6.5 it follows that $\lim_{m \rightarrow \infty} \Psi^{(m)}(\mathbf{x}, \xi_\mu^{(m)}) = \Psi^{(\infty)}(\mathbf{x}, \xi_\mu^{(\infty)})$ pointwise for $\mathbf{x} \in \text{Int}(\mathbf{A})$ and that $\lim_{m \rightarrow \infty} E^{(m)}(\xi_\mu^{(m)}) = E^{(\infty)}(\xi_\mu^{(\infty)})$. In other words, for $m \rightarrow \infty$ the eigenfunctions, the eigenvalues, and the eigenvalue equation for $H^{(m)}$ in Eq. (6.19b) converge pointwise to the eigenfunctions, the eigenvalues, and the eigenvalue equation for $H^{(\infty)}$ in Eq. (6.19a), respectively.

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